Asian-European Journal of Mathematics
Vol. 1, No. 1 (2010) 1–8

DOI: 10.1142/DOI Number

On the non-cyclic norm in non-periodic groups

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Received (January 14, 2019)
Revised (Day Month Year)

The authors study non-periodic locally soluble by finite groups with the non-Dedekind norm of non-cyclic subgroups, which is the intersection of normalizes of all non-cyclic subgroups of a group. It is found that all non-cyclic subgroups are normal in these groups. Their structure is described.

Keywords: non-periodic group; non-Dedekind group; non-cyclic subgroup; norm of group; norm of non-cyclic subgroups of a group.

AMS Subject Classification: 20E34, 20E36

1. Introduction

In the group theory findings, related to the study of groups, in which all subgroups of some non-empty system Σ are normal, are in the focus. It is clear that each group is an intersection of normalizers of all subgroups from Σ. If the system Σ contains all subgroups with some theoretical group property of a group G, then this intersection is called the Σ-norm of a group G and is denoted by $N_\Sigma(G)$.

Considering the Σ-norm, there are several problems related to the study of the properties of a group with the given system Σ of subgroups and some restrictions, which the Σ-norm satisfies. Many algebraists solved the similar problems but in the most of researches the whole group performed as the $N_\Sigma(G)$-norm. For the first time the situation when the Σ-norm is an own subgroup of the group was studied by R. Baer [1] for the system Σ, which consists of all subgroups of a group.

According to [1], the intersection of the normalizers of all subgroups of G is called the norm $N(G)$ of a group G. This direction of the research has found fairly widespread not only in the group theory, as evidenced by findings related to generalized norms of groups for systems of subgroups with different theoretical group properties, but also in the ring theory while studying Baer-kernel, which is a ring analogue of the norm of a group [2], and also in the vector space theory, while studying linear groups with finite dimensional orbits of the norm of the vector subspace [3].

In this article the research of the relations between the properties of a group and its Σ-norm ([4], [5]) is continued for the system Σ of all non-cyclic subgroups of a group. Such a Σ-norm of G is called the non-cyclic norm and is denoted by N_G [4]. It is evident, that the norm N_G is a characteristic subgroup of a group G and contains the center of a group.

If a group G is non-cyclic and coincides with the non-cyclic norm N_G, then all non-cyclic subgroups are normal in it. Non-Abelian locally solvable groups of this kind were studied by F. M. Lyman ([6], [7]) and were called $\overline{H}$-groups.

The following statement describes the structure of non-periodic locally solvable $\overline{H}$-groups.

Proposition 1.1. [6] *Non-periodic locally solvable $\overline{H}$-groups are the groups of the following types:*

1. $G = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = p^n$, $n \geq 1$ *($n > 1$ if $p = 2$)*, $|b| = \infty, [a, b] = a^{p^{n-1}}$;
2. $G = H \times B$, *where* $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$, B *is an infinite cyclic group or a group isomorphic to an additive group of dyadic numbers.*

Thus, the non-cyclic norm N_G of non-periodic group G can be Dedekind (Abelian or Hamiltonian) subgroup or $\overline{H}$-group.

2. Preliminary Results

In further researches the following statement is widely used.

Lemma 2.1. *The non-cyclic norm N_G of a non-periodic locally soluble by finite group G is Dedekind in each of the following cases:*

1. *a group G contains a non-cyclic subgroup H, such that $H \bigcap N_G = E$;*
2. *the non-cyclic norm N_G of a group G is periodic;*
3. *a group G contains a cyclic N_G-admissible subgroup $\langle g \rangle$, such that $\langle g \rangle \bigcap N_G = E$;*
4. *a group G contains a free Abelian subgroup of rank 2.*

Proof. (1) Let N_G be the non-cyclic norm of a group G and H be a non-cyclic subgroup, such that $H \bigcap N_G = E$. Then $[H, \langle y \rangle] \subseteq H \bigcap N_G = E$ for an arbitrary element $y \in N_G$. Since the subgroup $\langle H, y \rangle = H \times \langle y \rangle$ is non-cyclic, it is N_G-admissible. But in this case

$$(H \times \langle y \rangle) \bigcap N_G = \langle y \rangle \triangleleft N_G$$

and the norm N_G is Dedekind.

(2) Let N_G be a periodic non-Dedekind group. Let us take an element $x \in G$, such that $|x| = \infty$ and let us consider the group $G_1 = \langle x \rangle N_G$. The norm N_G

is a periodic non-Hamiltonian $\overline{H}$-group (see [6]). It contains a finite characteristic subgroup M by the description of such groups. Then $M \lhd G_1$, $C_{G_1}(M) \lhd G_1$ and $[G_1 : C_{G_1}(M)] < \infty$. Therefore, there is an element $x_1 \in \langle x \rangle, |x_1| = \infty$ for which $[\langle x_1 \rangle, M] = E$. But then $\langle x_1, M \rangle$ is non-cyclic and, therefore, N_G-admissible.

Let $|M| = m$. Then the subgroup $\langle x_1^m \rangle$ is N_G-admissible. Since $[\langle x_1^m \rangle, N_G] \subseteq \langle x_1^m \rangle \bigcap N_G = E$, the subgroup $\langle a, x_1^m \rangle$ is Abelian non-cyclic for any element $a \in N_G$. So it is N_G-admissible. But then

$$\langle a \rangle = \bigcap_{k=1}^{\infty} \langle a, x_1^{km} \rangle \lhd N_G$$

for an arbitrary positive integer k and N_G is Dedekind.

(3) Let $\langle g \rangle$ be a cyclic subgroup which satisfies the condition (3) of the Lemma. Suppose that the norm N_G is non-Dedekind. It is non-periodic by the proved above. Therefore, it is a non-periodic $\overline{H}$-group of one of the types (1)-(2) of Proposition 1.1.

Since

$$[\langle g \rangle, N_G] \subseteq \langle g \rangle \bigcap N_G = E,$$

the subgroup $\langle g, x \rangle$ is Abelian non-cyclic for an arbitrary element $x \in N_G$, $|x| = \infty$. So

$$\langle g, x \rangle \bigcap N_G = \langle x \rangle \lhd N_G$$

and all infinite cyclic subgroups of the norm N_G are normal in it, which contradicts the properties of the subgroup N_G.

(4) Suppose that a group G contains a free Abelian subgroup $A = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = \infty$ and its non-cyclic norm N_G is non-Dedekind. Then N_G is a non-periodic $\overline{H}$-group by the proved above. So it is a group of one of the types (1)-(2) of Proposition 1.1 [6]. Since the norm N_G does not contain free Abelian subgroups of rank 2, $\langle x \rangle \bigcap N_G = E$.

Since the subgroup $\langle x, y^k \rangle$ is N_G-admissible for an arbitrary positive integer k, the subgroup

$$\langle x \rangle = \bigcap_{k=1}^{\infty} \langle x, y^k \rangle$$

is also N_G-admissible. The subgroup N_G is Dedekind by the condition (3) of the Theorem. Lemma is proved. □

Corollary 2.1. *If the norm N_G of non-cyclic subgroups of a non-periodic locally soluble by finite group G is non-Dedekind, then every non-cyclic subgroup and every cyclic subgroup, normal in G, has a non-identity intersection with the norm N_G.*

Lemma 2.2. *If a non-periodic locally soluble by finite group G has the non-Dedekind norm N_G of non-cyclic subgroups, then the center $Z(G)$ of a group G contains elements of infinite order.*

Proof. Let G have the non-Dedekind non-cyclic norm N_G. Then N_G is a non-periodic $\overline{H}$-group by Lemma 2.1 and is a group of one of the types (1)-(2) of Proposition 1.1.

Let us denote by Z a power of the center $Z(N_G)$, which does not contain elements of finite order, different from unit. Then Z is an infinite cyclic group or a group isomorphic to the additive group of dyadic numbers. Let us prove that $Z \subseteq Z(G)$.

Let $x \in G$, $|x| < \infty$. Then $\langle x, Z^{p^k}\rangle \lhd G_1 = \langle x\rangle N_G$ for a prime number $p \neq 2$ and an arbitrary positive integer k. Therefore,

$$\bigcap_{k=1}^{\infty} \left\langle Z^{p^k}, x \right\rangle = \langle x\rangle \lhd G_1$$

and $[\langle x\rangle, Z] \subseteq Z \bigcap \langle x\rangle = E$.

Let us prove that the subgroup Z centralizes every element $x \in G$ of infinite order. Assume that $\langle x\rangle \bigcap Z = E$. Then the subgroups $Z \leftthreetimes \langle x\rangle$ and $T(N_G) \leftthreetimes \langle x\rangle$, where $T(N_G)$ is the torsion part of non-cyclic norm N_G, are non-cyclic because $Z \lhd G$ and $T(N_G) \lhd G$. Therefore, these subgroups are N_G-admissible. So, $\langle x\rangle = (Z \leftthreetimes \langle x\rangle) \cap (T(N_G) \leftthreetimes \langle x\rangle)$ is also N_G-admissible and $[N_G, \langle x\rangle] = E$. But in this case the group $G_1 = N_G\langle x\rangle$ contains a free Abelian subgroup of rank 2 and N_G is Dedekind by Lemma 2.1, which is impossible. Thus, $\langle x\rangle \bigcap Z \neq E$.

Let $z \in Z$ and $x^{-1}zx = z_1, z_1 \in Z$. Since $\langle x\rangle \bigcap Z \neq E$, $z^n \in \langle x\rangle$ for some positive integer n, because the subgroup Z is locally cyclic. Then $x^{-1}z^n x = z_1^n = z^n$, $z = z_1$ and $[x, z] = 1$. Therefore, $Z \subseteq Z(G)$. The Lemma is proved. □

Corollary 2.2. *A non-periodic locally soluble by finite group G with non-Dedekind non-cyclic norm N_G does not contain finite Abelian non-cyclic subgroups.*

Proof. Let the non-cyclic norm N_G of a group G be non-Dedekind. Then it is a non-periodic $\overline{H}$-group of one of the types (1)-(2) of Proposition 1.1 by Lemma 2.1. Suppose that the group G contains a finite Abelian non-cyclic subgroup A.

According to Lemma 2.2 the center $Z(G)$ of the group G is non-periodic. Then for an arbitrary element $a \in A$, $a \neq 1$ the subgroup $\langle a, z\rangle$, where $z \in Z(G)$, $|z| = \infty$, is non-cyclic. So $\langle a, z\rangle$ is N_G-admissible. Hence, the subgroup $\langle a\rangle$ is also N_G-admissible. Assume that $A \nsubseteq N_G$. Then there exists an element $x \in A$ such that $\langle x\rangle \bigcap N_G = E$.

The subgroup $\langle x\rangle$ is N_G-admissible by the proved above. The non-cyclic norm N_G is Dedekind by Lemma 2.1, which contradicts the condition of the Corollary. □

3. Non-periodic locally soluble by finite groups with the non-Dedekind norm of non-cyclic subgroups

In [4] it was proved that a non-periodic locally soluble by finite group, which non-cyclic norm N_G has a finite index, is a finite extension of the center $Z(G)$. And the norm N_G is Abelian if $1 < |G/N_G| < \infty$. Let's consider non-periodic locally solvable by finite groups with the non-Dedekind norm N_G without restrictions on its index. Their structure is completely characterized by the following Theorem, which summarizes the results of [4] and [7].

Theorem 3.1. *A non-Abelian non-periodic locally soluble by finite group G has the non-Dedekind non-cyclic norm N_G, if and only if all non-cyclic subgroups are normal in G and $G = N_G$.*

Proof. The sufficiency of the condition of the Theorem does not require proof, so we need only to prove the necessity of the Theorem.

Suppose that the norm N_G contains all elements of infinite order of a group G. Let us show that in this case it also contains all elements of finite order and $G = N_G$.

Let $x \in G$ be an arbitrary element of finite order. The center $Z(G)$ of a group G contains elements of infinite order by Lemma 2.2. Let $z \in Z(G)$, $|z| = \infty$, then $|zx| = \infty$. Hence, $zx \in N_G$ by assumption and $x \in N_G$. Thus, in this case $G = N_G$.

Let us consider that the group G contains elements of infinite order which do not belong to the norm N_G. Let us show that in this case the quotient group G/N_G is periodic. Suppose that it is false. Then there is an element $x \in G \backslash N_G$ of infinite order such that $\langle x \rangle \bigcap N_G = E$.

Let us consider the group $G_1 = N_G \leftthreetimes \langle x \rangle$. Since the norm N_G is non-Dedekind, it is non-periodic by Lemma 2.1 and a $\overline{H}$-group of one of two types of Proposition 1.1.

It is evident that in both cases the derived subgroup $N_G' = M$ of the norm N_G is finite. Therefore, $[G_1 : C_{G_1}(M)] < \infty$ for some number $k \in N$ and $x^k \in C_{G_1}(M)$. Since the subgroup $\langle x^k, M \rangle$ is non-cyclic, it is N_G-admissible and the subgroup $\langle x^{mk} \rangle$, where $|M| = m$, is also N_G-admissible. The norm N_G is Dedekind by Lemma 2.1, which contradicts the condition. Thus, the quotient group G/N_G is periodic. Taking into account that G is locally solvable by finite, the quotient group G/N_G is locally finite.

Let us denote by Z the smallest natural power of the center $Z(N_G)$, which does not contain elements of finite order, different from the unity. So, $Z \subseteq Z(G)$ by the proof of Lemma 2.2.

Considering that the quotient group G/Z is locally finite and by Proposition 3.9 [8], we conclude that the torsion part $T(G)$ of the group G is locally finite and the quotient group $G/T(G)$ is an Abelian torsion-free group. G does not contain a free Abelian subgroup of rank 2 by Lemma 2.1. Thus, $G/T(G)$ is an Abelian torsion-free group of rank 1.

Let us consider two cases, depending on the structure of the group N_G.

1) Let the group G has the non-cyclic norm N_G of the type (1):

$$N_G = H \times B,$$

where $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$, B is an infinite cyclic group or a group isomorphic to an additive group of dyadic numbers.

Let us show that all elements of finite order of the group G are contained in the non-cyclic norm N_G and $T(G) = T(N_G)$.

Suppose that it is false. Let us consider the subgroup $G_1 = N_G \langle x \rangle$, where $x \in T(G) \backslash T(N_G)$. Since $[G_1 : N_{G_1}] < \infty$ and the subgroup N_G is non-Dedekind, $G_1 = N_{G_1}$ by Theorem 4 [4], which contradicts the structure of non-periodic $\overline{H}$-groups. Therefore, $T(G) = T(N_G) = H$, where H is a quaternion group.

Let us consider the commutator $[G, H]$. If $x \in H$, then $[\langle x \rangle, H] \subseteq \langle h_1^2 \rangle$. Let $x \in G \backslash H$. Then $\langle x, h_1^2 \rangle$ is a non-cyclic subgroup, and, therefore,

$$[\langle x \rangle, H] \subseteq H \bigcap \langle x, h_1^2 \rangle = \langle h_1^2 \rangle.$$

So $[G, H] \subseteq \langle h_1^2 \rangle$ and $\langle h \rangle \lhd G$ for an arbitrary element $h \in H$. Thus,

$$[G : C_G(\langle h_1 \rangle)] = [G : C_G(\langle h_2 \rangle)] = 2,$$

$[G : C_G(H)] = 4$ and $G = H \cdot C_G(H)$.

Let us consider that G/H is a locally cyclic group and

$$G/H = HC_G(H)/H \cong C_G(H)/H \bigcap C_G(H) = C_G(H)/\langle h_1^2 \rangle,$$

we conclude that the group $C_G(H)/\langle h_1^2 \rangle$ is also locally cyclic.

Let us show that the preimage of this group is Abelian. Let $x, y \in C_G(H)$, $|x| = |y| = \infty$. Since the quotient group $\langle x, y, h_1^2 \rangle \langle h_1^2 \rangle$ is cyclic, the subgroup $\langle x, y, h_1^2 \rangle$ is Abelian and $[x, y] = 1$. Thus, the subgroup $C_G(H)$ is Abelian and $C_G(H) = \langle h_1^2 \rangle \times C$, where C is an Abelian torsion-free group of rank 1. But in this case $C \subseteq Z(G), [G : Z(G)] < \infty$ and $G = N_G$ by the results of [4], which is the desired conclusion.

2) Let the non-cyclic norm N_G be a $\overline{H}$-group of the type (2):

$$N_G = \langle a \rangle \leftthreetimes \langle b \rangle, |a| = p^n, n \geq 1, |b| = \infty, [a, b] = a^{p^{n-1}}, n > 1, if p = 2.$$

The torsion part $T(G)$ of the group G is locally finite by the proved above. Suppose that $T(G)$ contains an element x of a prime order, which does not belong to the norm N_G. The center $Z(G)$ of the group G is non-periodic by Lemma 2.2. Then $\langle x, z^k \rangle \lhd G_1 = \langle x \rangle N_G$ for $z \in Z(G)$, $|z| = \infty$ and an arbitrary positive integer k. Therefore,

$$\bigcap_{k=1}^{\infty} \left\langle z^k, x \right\rangle = \langle x \rangle \lhd G_1.$$

It means that the subgroup $\langle x \rangle$ is N_G-admissible, which contradicts Lemma 2.1. Thus, $T(G)$ is a primary locally finite group which has an unique subgroup of prime order.

Suppose that $|T(G)| = \infty$. Then $T(G)$ is a finite extension of a quasicyclic subgroup A and $A \lhd G$ as a characteristic subgroup of $T(G)$.

Since $[A, N_G] \subseteq A \bigcap N_G \subseteq \langle a \rangle$, where $\langle a \rangle \in T(N_G)$, $[\langle a \rangle, N_G] = E$, which is impossible. Thus, $|T(G)| < \infty$ and $T(G)$ is a primary cyclic group or a generalized quaternion group.

Let $T(G) = \langle g \rangle$ be a cyclic subgroup. If $|g| = p^k > |a|$, then

$$[b, zg] = [b, g] \subseteq \left\langle g^{p^{k-1}}, gz \right\rangle \bigcap \langle g \rangle = \left\langle g^{p^{k-1}} \right\rangle$$

for elements $b \in N_G$, $z \in Z(G)$, $|z| = \infty$. It means that $[b, g^p] = 1$ and so $[b, a] = 1$, which contradicts the condition. Thus,

$$T(G) = T(N_G) = \langle a \rangle .$$

Let us prove that the subgroup $C = C_G(\langle a \rangle)$ is Abelian. In fact, if $x, y \in C$, then the quotient group $\langle x, y, a \rangle / \langle a \rangle$ is cyclic by the locally cyclicity of the quotient group $G/T(G) = G/\langle a \rangle$. Therefore, $\langle x, y, a \rangle$ is Abelian and $[x, y] = 1$. The subgroup C is also Abelian by the arbitrariness of the choice of the elements x and y.

Thus, $C = \langle a \rangle \times C_1$, where C_1 is an Abelian torsion-free group of rank 1. Suppose that the subgroup C_1 is non-cyclic. Then the subgroup $C_1^{p^n}$ $(n \in N)$ is also non-cyclic and

$$[y, a] \in \langle a \rangle \bigcap \langle y, C_1^{p^n} \rangle = E,$$

for an arbitrary element $y \in G \backslash C$, contrary to its choice.

It follows that $C_1 = \langle c \rangle$ is an infinite cyclic group. Taking into account that $[G : \langle c \rangle] < \infty$ and $\langle b \rangle \bigcap \langle c \rangle \neq E$, we conclude that $[c, b] = 1$, hence $[G : Z(G)] < \infty$. $G = N_G$ by the results of [4].

Now let us consider the case when the torsion part $T(G)$ of the group G is a generalized quaternion group:

$$T(G) = \langle h_1, h_2 \rangle, |h_1| = 2^k \geq 4, |h_2| = 4, h_2^2 = h_1^{2^{k-1}}, h_2^{-1} h_1 h_2 = h_1^{-1}.$$

Since $\langle a \rangle = T(N_G) \lhd G$, without loss of generality we can assume that $a \in \langle h_1 \rangle$. Let h be an arbitrary element of order 4 from $T(G)$ and $z \in Z(G)$, $|z| = \infty$. Then $\langle h, z^m \rangle \lhd G_1 = \langle h \rangle N_G$ and

$$\langle h \rangle = \bigcap_{k=1}^{\infty} \langle z^m, h \rangle \lhd G_1$$

for a positive integer m. So $T(G)$ is a quaternion group and $|a| = 4$.

Let us denote $C = C_G(a)$. Since $[G : C] = 2$, $C = T(G) \cdot C$. The group C is Abelian because the quotient group

$$G/T(G) \cong C/C \bigcap T(G) = C/\langle a^2 \rangle$$

is locally finite and by the proved above. But in such case $[G : Z(G)] < \infty$ and the group G coincides with the norm N_G [4], which is impossible. The Theorem is proved. □

Let us formulate some important corollaries of the Theorem.

Corollary 3.1. *If the non-cyclic norm N_G of a non-periodic locally soluble by finite group G is non-Abelian, then either N_G is a quaternion group or $N_G = G$ and G is a $\overline{H}$-group.*

Corollary 3.2. *If a non-periodic locally soluble by finite involution-free group G contains a non-invariant non-cyclic subgroup, then its non-cyclic norm N_G is Abelian.*

Corollary 3.3. *A non-periodic locally soluble by finite group, which has the non-Dedekind non-cyclic norm, is soluble of degree 2.*

The following example confirms the existence of non-periodic locally soluble by finite groups, whose non-cyclic norm is a quaternion group.

Example 3.1. $G = Q \times (\langle x \rangle \leftthreetimes \langle b \rangle)$, $Q = \langle h_1, h_2 \rangle$ is a quaternion group, $|x| = \infty$, $|b| = 2$, $b^{-1}xb = x^{-1}$.

In this case $N_G = Q$. In fact,

$$N_G \subseteq N_G(\langle h_1, bx \rangle) \bigcap N_G(\langle h_1, b \rangle) = (Q \times \langle bx \rangle) \bigcap (Q \times \langle b \rangle) = Q.$$

The subgroup Q normalizes every non-cyclic subgroup H from G. It is evident, if $H \subseteq \langle b, x \rangle$. If $H \not\subset \langle b, x \rangle$, then H contains at least one element of the form hx_i or hx_ib, where $h \in Q$. So in any cases $H \supset \left\langle h^2 \right\rangle$ and $N_G = Q$.

4. Acknowledgment

The author would like to thank the referee for their very useful comments which improved the paper.

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