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НЕРІВНОСТІ ГЕЛЬДЕРА ТА МІНКОВСЬКОГО ТА ЇХНІ УЗАГАЛЬНЕННЯ

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АНОТАЦІЯ

Постановка задачі. *Велика кількість математичної літератури присвячена класичним нерівностям. Нерівності Гельдера, окремим випадком яких є нерівність Коші-Буняковського, а також нерівність Мінковського, що є нерівністю многокутника в нормованому просторі, які лежать в основі геометрії унітарних та нормованих просторів – скінченних та нескінченновимірних. У статті розглядається узагальнення цих конструкцій – як у дискретній формі, тобто для скінченних сум та рядів, так і для інтегралів. Істотно, що нерівності для сум доводяться елементарними методами, без використання диференціального числення. Отримані результати можуть бути використані в науковій діяльності для обчислення деяких виразів у вигляді сум або інтегралів, а також учнями при підготовці до олімпіад і навіть для вивчення математики в шкільних гуртках.*

Матеріали і методи. *Для доведення узагальненої нерівності Мінковського та інтегральних нерівностей Гельдера та Мінковського використана узагальнена нерівність Гельдера для сум, яка була раніше отримана автором і, у свою чергу, була виведена з нерівності Коші.*

Результати. *Було доведено узагальнені нерівності Мінковського – для скінченних сум та нескінченних рядів з невід'ємними членами та інтеграл для невід'ємних функцій, а також узагальнену інтегральну нерівність Гельдера та, в окремому випадку, нерівність Коші-Буняковського.*

Висновки. *Застосування узагальнених нерівностей Гельдера та Мінковського для сум, рядів та інтегралів є досить ефективним методом, який дозволяє отримати цікаві наслідки, важливі оцінки – потрібно лише успішно вибрати скінченновимірні або нескінченновимірні вектори чи функції та застосувати до них доведені нерівності. На цьому шляху є великий простір для творчої діяльності.*

КЛЮЧОВІ СЛОВА: *нерівність Гельдера; нерівність Мінковського; лінійний простір; норма; інтегральна нерівність.*

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THE INEQUALITIES OF HELDER AND MINKOVSKY AND THEIR GENERALIZATIONS

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ABSTRACT

Formulation of the Problem. *A large amount of mathematical literature is devoted to classical inequalities. Helder's inequalities, a special case of which is the Cauchy-Buniakovsky inequality, as well as Minkowski's, which is a polygon inequality in a normed space, underlie the geometry of unitary and normed spaces - finite and infinite-dimensional (Banach). The article considers the generalization of these constructions - both in discrete form, that is, for finite sums and series, and for integrals. It is essential that inequalities for sums are proved by elementary methods, without the use of differential calculus. The results obtained can be used in scientific activities for evaluating some expressions in the form of sums or integrals, as well as by students in preparation for Olympiads and even for studying mathematics in school circles.*

Materials and Methods. *To prove the generalized Minkowski inequality and the integral inequalities of Helder and Minkowski, the generalized Helder inequality for sums, which was previously obtained by the author which, in turn, was derived from Cauchy's inequality.*

Results. *The generalized Minkowski inequalities were proved for finite sums and infinite series with non-negative members and the integral for non-negative functions, as well as the generalized integral Helder inequality and, in a special case, the Cauchy-Bunyakovsky inequality.*

Conclusion. *The application of the generalized Helder and Minkowski inequalities for sums, series, and integrals is a fairly effective method that allows you to obtain interesting consequences, important estimates – you only need to successfully select finite-dimensional or infinite-dimensional vectors or functions and apply the proved inequalities to them. On this path, there is a great deal of space for creative activity.*

KEYWORDS: *Helder's inequality; Minkowski's inequality; linear space; norm; integral inequality.*

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INTRODUCTION

Problem statement. Proving inequalities is one of the important topics in the school course, especially in educational institutions with an in-depth study of mathematics. Some similar problems are solved using classical inequalities, in particular, Cauchy, Cauchy-Bunyakovsky, Helder (Hölder), Minkowski, etc. It is interesting to prove which of these inequalities can be proved without using differential calculus methods, using elementary methods, limited in means, as a rule, require a creative approach, which is very valuable, especially in the education of the future mathematical elite. The beginner develops skills for creative, independent work.

Analysis of current research. Numerous mathematical literature is devoted to classical inequalities and their consequences. The author develops the concept of previous works (Bokhonov, 2022; Bokhonov & Bokhonova, 2023): some inequalities can be proved using known inequalities, usually classical or their generalizations, by selecting in a certain way successful replacements of the modified ones, after which the considered inequality becomes a partial case. On this path, the authors managed, in particular, to prove the classical inequality for medium degrees, which is a general fact and application, for example, in probability theory. They also managed to solve several Olympiad problems using the general Cauchy-Buniakovsky inequality. The application of classical inequalities in the educational process in the course of mathematical analysis and their generalization (unfortunately, without proofs) is devoted to the article (Zhuravska & Shramenko, 2010), as well as D. S. Mitrinovic & J. E. Pecaric (1993). The article (Martynenko & Chkana, 2017) is devoted to the proof of inequalities using methods of mathematical analysis. A review of results related to classical inequalities can be found in (Steele, 2004). On the other hand, for example, the Cauchy-Bunyakovsky inequality suggests a significant generalization – transfer to C*-modules (Aldaz et al., 2015).

The purpose of the work is to generalize the classical Hölder and Minkowski inequalities to the infinite-dimensional case, reduce them to integrals, and demonstrate the obtained results on the example of specific problems.

RESEARCH METHODS

The study is based on the analysis of both school mathematics course programs and mathematical analysis for first-year undergraduate students of technical higher education institutions and mechanical and mathematical faculties of universities. The results of the cited previous works of the author are applied.

RESEARCH RESULTS

Let us proceed to the formulation and consideration of the problem.

Let be $A_k = \left(a_{1k}, a_{2k}, \dots, a_{nk}\right)^T, a_{jk} \ge 0, k = 1,\dots,m,\ j = 1,\dots,n$ - a system of vectors of a linear space $\mathbb{R}^n$ (“*T*” stands for transpose). You can give up on inalienability a_{kj} and work with vectors whose coordinates are the moduls of these numbers.

Next, norms of the form will be used $\|x\|_p = \left(\sum_{j=1}^{n} |x|_j^p\right)^{\frac{1}{p}}$.

One of the tasks is to generalize the classical Minkowski inequality for $m \ge 2$ column vectors:

$$\left(\left(a_{11}+a_{12}+\dots+a_{1m}\right)^p+\dots+\left(a_{n1}+a_{n2}+\dots+a_{nm}\right)^p\right)^{\frac{1}{p}} \le \left(a_{11}^p+\dots+a_{n1}^p\right)^{\frac{1}{p}}+\dots+\left(a_{1m}^p+\dots+a_{nm}^p\right)^{\frac{1}{p}},\ p>1 \tag{1}$$

The second problem is to transfer the generalized Hölder inequality, proven by the author in (Bokhonov, 2022; Bokhonov & Bokhonova, 2023) for finite-dimensional vectors, to the infinite-dimensional case:

$$\forall p_k > 1, k = 1,\dots,m,\ \frac{1}{p_1}+\dots+\frac{1}{p_m} = 1 \tag{2}$$

$$\sum_{n=1}^{\infty} a_{n1}a_{n2}\dots a_{nm} \le \left(\sum_{n=1}^{\infty}\left(a_{n1}^{p_1}\right)\right)^{\frac{1}{p_1}}\dots\left(\sum_{n=1}^{\infty}\left(a_{nm}^{p_m}\right)\right)^{\frac{1}{p_m}}. \tag{3}$$

The third problem is to prove the generalized integral inequality of Hölder. Let $f_1,\dots,f_m$ – continuous non-negative in $[a,b]$ functions. Then, if condition (2) is satisfied and the following integrals converge,

$$\left(\int_a^b \left(f_1(x)\right)^{p_j} dx\right)^{\frac{1}{p_j}} < \infty,\ j = 1,\dots,m \tag{4}$$

the inequality (generalized Hölder) takes place:

$$\int_a^b f_1(x)\dots f_m(x)dx \le \left(\int_a^b \left(f_1(x)\right)^{p_1} dx\right)^{\frac{1}{p_1}}\dots\left(\int_a^b \left(f_m(x)\right)^{p_m} dx\right)^{\frac{1}{p_m}}. \tag{5}$$

On condition $p_1 = \dots = p_m \equiv p$ we obtain the generalized Cauchy-Buniakovsky inequality.

The fourth problem is to prove the generalized Minkowski integral inequality for $p > 1$:

$$\left(\int_a^b \left(f_1(x)+\dots+f_m(x)\right)^p dx\right)^{\frac{1}{p}} \le \left(\int_a^b \left(f_1(x)\right)^p dx\right)^{\frac{1}{p}}+\dots+\left(\int_a^b \left(f_m(x)\right)^p dx\right)^{\frac{1}{p}}. \tag{6}$$

The inequalities of Cauchy-Bunyakovsky, Hölder and some others are stated, for example, in the cited works. At the same time, the methods of mathematical analysis are used for the proof in most of them. The purpose of the proposed article is to prove inequalities (3) for sums by elementary methods, without the use of differential calculus, which allows them to be studied by high school students. Then they are transferred to integrals, which allows the results to be used in preparation for student olympiads.

II. Proving inequalities

1. Generalized Minkowski inequality. For $p, q > 1,\ \frac{1}{p}+\frac{1}{q} = 1$, using Hölder's inequality, we obtain:

$$\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}=\sum_{i=1}^{n}a_{i1}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p-1}+\ldots+\sum_{i=1}^{n}a_{im}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p-1}\le$$

$$\le\left(\sum_{i=1}^{n}\left(a_{i1}\right)^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{(p-1)q}\right)^{\frac{1}{q}}+\ldots+\left(\sum_{i=1}^{n}\left(a_{im}\right)^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{(p-1)q}\right)^{\frac{1}{q}}=$$

$$=\left(\left(\sum_{i=1}^{n}\left(a_{i1}\right)^{p}\right)^{\frac{1}{p}}+\ldots+\left(\sum_{i=1}^{n}\left(a_{im}\right)^{p}\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{(p-1)q}\right)^{\frac{1}{q}}.$$

Noticing, that $1-\frac{1}{q}=\frac{1}{p}$, $q(p-1)=p$, dividing both parts on

$\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{(p-1)q}\right)^{\frac{1}{q}}=\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{\frac{1}{q}}$, we obtain on the left-hand side:

$$\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{1-\frac{1}{q}}=\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{\frac{1}{p}}$$

and as a result we obtain the generalized Minkovski inequality:

$$\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{\frac{1}{p}}\le\left(\left(\sum_{i=1}^{n}\left(a_{i1}\right)^{p}\right)^{\frac{1}{p}}+\ldots+\left(\sum_{i=1}^{n}\left(a_{im}\right)^{p}\right)^{\frac{1}{p}}\right)$$

or

$$\left\|A_1+A_2+\ldots+A_m\right\|_p\le\left\|A_1\right\|_p+\left\|A_2\right\|_p+\ldots+\left\|A_m\right\|_p .$$

2. Consider infinite-dimensional vectors $A_k=\left(a_{1k},a_{2k},\ldots,a_{nk},\ldots\right)^{T}\in l_{p_k}$, $k=1,\ldots,m$. That is, the series converges $\left\|A_k\right\|_k^{p_k}=\sum_{i=1}^{\infty}\left(a_{ik}\right)^{p_k}<\infty, k=1,\ldots,m, a_{i,k}\ge 0$. This allows us to move to the limit in the generalized Hölder inequality, proved in (Bokhonov & Bokhonova, 2023) for finite-dimensional vectors under condition (2)

$$\sum_{n=1}^{n}a_{n1}a_{n2}\ldots a_{nm}\le\left(\sum_{n=1}^{n}\left(a_{n1}^{p_1}\right)\right)^{\frac{1}{p_1}}\ldots\left(\sum_{n=1}^{n}\left(a_{nm}^{p_m}\right)\right)^{\frac{1}{p_m}}$$

and obtain inequality (3) for infinite-dimensional vectors.
Similarly, the generalized Minkowski inequality in the infinite-dimensional case is proved:

$$\left(\sum_{i=1}^{\infty}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{\frac{1}{p}}\le\left(\left(\sum_{i=1}^{\infty}\left(a_{i1}\right)^{p}\right)^{\frac{1}{p}}+\ldots+\left(\sum_{i=1}^{\infty}\left(a_{im}\right)^{p}\right)^{\frac{1}{p}}\right).$$

3. Proving the generalized integral inequality of Hölder.
Divide the segment $\left[a,b\right]$ with points $a=x_0<x_1<\ldots<x_n=b$ and, bearing in mind (2), we introduce the notation:

$$\forall i=1,\ldots,n \; a_{i1}=f_1\left(x_i\right)\Delta x_i^{\frac{1}{p_1}},\ldots,a_{im}=f_m\left(x_i\right)\Delta x_i^{\frac{1}{p_m}} .$$

Let's use the generalized Hölder inequality:

$$\sum_{i=1}^{n}a_{i1}a_{i2}\ldots a_{im}=\sum_{i=1}^{n}f_1\left(x_i\right)\ldots f_m\left(x_i\right)\Delta x_i\le\left(\sum_{n=1}^{n}\left(a_{n1}^{p_1}\right)\right)^{\frac{1}{p_1}}\ldots\left(\sum_{n=1}^{n}\left(a_{nm}^{p_m}\right)\right)^{\frac{1}{p_m}}\le$$

$$\le\left(\sum_{i=1}^{n}\left(f_1\left(x_i\right)\right)_{i1}^{p_1}\Delta x_i\right)^{\frac{1}{p_1}}\ldots\left(\sum_{i=1}^{n}\left(f_m\left(x_i\right)\right)_{im}^{p_m}\Delta x_i\right)^{\frac{1}{p_m}}.$$

Going to the limit in the proved inequality when $\max\Delta x_i\to 0$, we obtain inequality (5), which we had to prove.

$$\int_a^b f_1(x)\ldots f_m(x)dx\le\left(\int_a^b\left(f_1(x)\right)^{p_1}dx\right)^{\frac{1}{p_1}}\ldots\left(\int_a^b\left(f_m(x)\right)^{p_m}dx\right)^{\frac{1}{p_m}}$$

4. Proving the generalized integral inequality of Minkovsky.
Let us introduce the notation:

$$\forall i=1,\ldots,n \; a_{i1}=f_1\left(x_i\right)\Delta x_i^{\frac{1}{p}},\ldots,a_{im}=f_m\left(x_i\right)\Delta x_i^{\frac{1}{p}}, p>1 .$$

Let's use the generalized Minkowski inequality for sums:

$$\left(\sum_{i=1}^{n}\left(a_{i1}+a_{i2}+\ldots+a_{im}\right)^{p}\right)^{\frac{1}{p}}=\left(\sum_{i=1}^{n}\left(f_1\left(x_i\right)+\ldots+f_m\left(x_i\right)\right)^{p}\Delta x_i\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{n}\left(a_{i1}\right)^{p}\right)^{\frac{1}{p}}+\ldots+\left(\sum_{i=1}^{n}\left(a_{im}\right)^{p}\right)^{\frac{1}{p}}=$$

$$\le \left(\sum_{i=1}^{n} \left(f_1\left(x_i\right)\right)_{i1}^{p} \Delta x_i \right)^{\frac{1}{p}} + \ldots + \left(\sum_{i=1}^{n} \left(f_m\left(x_i\right)\right)_{im}^{p} \Delta x_i \right)^{\frac{1}{p}} .$$

Passing to the limit at in the proved inequality, we obtain inequality (6), which we had to prove:

$$\left(\int_a^b \left(f_1(x) + \ldots + f_m(x)\right)^p dx \right)^{\frac{1}{p}} \le \left(\int_a^b \left(f_1(x)\right)^p dx \right)^{\frac{1}{p}} + \ldots + \left(\int_a^b \left(f_m(x)\right)^p dx \right)^{\frac{1}{p}} .$$

III. Examples

We will show how to apply the obtained results to prove some inequalities. For an easier form of notation, we will use vectors of small dimensions. Obviously, similar results hold for vectors of arbitrary dimensions. The coordinates of the vectors are assumed to be non-negative, as before.

1. $A=\left(a,a^2,a^3\right)^T, B=\left(b,b^2,b^3\right)^T, C=\left(c,c^2,c^3\right)^T, \ p=3$. Let's apply the inequality (1):

$$\left(\left(a+b+c\right)^3 + \left(a^2+b^2+c^2\right)^3 + \left(a^3+b^3+c^3\right)^3 \right)^{\frac{1}{3}} \le \left(a^3+a^6+a^9\right)^{\frac{1}{3}} + \left(b^3+b^6+b^9\right)^{\frac{1}{3}} + \left(c^3+c^6+c^9\right)^{\frac{1}{3}} =$$

$$= a\left(1+a^3+a^6\right)^{\frac{1}{3}} + b\left(1+b^3+b^6\right)^{\frac{1}{3}} + c\left(1+c^3+c^6\right)^{\frac{1}{3}} .$$

2. $A_1=\left(\frac{a}{b},\frac{b}{a}\right)^T, A_2=\left(\frac{b}{c},\frac{c}{b}\right)^T, A_2=\left(\frac{c}{a},\frac{a}{c}\right)^T, p=3$. Let's apply the inequality (1):

$$\left(\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^3 + \left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3 \right)^{\frac{1}{3}} \le \left(\left(\frac{a}{b}\right)^3 + \left(\frac{b}{a}\right)^3 \right)^{\frac{1}{3}} + \left(\left(\frac{b}{c}\right)^3 + \left(\frac{c}{b}\right)^3 \right)^{\frac{1}{3}} + \left(\left(\frac{c}{a}\right)^3 + \left(\frac{a}{c}\right)^3 \right)^{\frac{1}{3}} .$$

3. $A=\left(a,a,b\right)^T, B=\left(b,b,c\right)^T, C=\left(c,c,a\right)^T, \ p=3$. Let's apply the inequality (1):

$$\left(\left(a+b+c\right)^3 + \left(a+b+c\right)^3 + \left(b+c+a\right)^3 \right)^{\frac{1}{3}} \le \left(2a^3+b^3\right)^{\frac{1}{3}} + \left(2b^3+c^3\right)^{\frac{1}{3}} + \left(2c^3+a^3\right)^{\frac{1}{3}} \Rightarrow$$

$$\sqrt[3]{3}\left(a+b+c\right) \le \left(2a^3+b^3\right)^{\frac{1}{3}} + \left(2b^3+c^3\right)^{\frac{1}{3}} + \left(2c^3+a^3\right)^{\frac{1}{3}} .$$

4. Evaluate the integral $\int_0^{\frac{\pi}{2}} \left|\sin x \sin 2x \ldots \sin 2mx\right| dx$.

We have: $\left(\int_0^{\frac{\pi}{2}} \left|\sin x \sin 2x \ldots \sin 2mx\right| dx \right) \le \left(\int_0^{\frac{\pi}{2}} \left(\sin x\right)^{2m} dx \right)^{\frac{1}{2m}} \left(\int_0^{\frac{\pi}{2}} \left(\sin 2x\right)^{2m} dx \right)^{\frac{1}{2m}} \ldots \left(\int_0^{\frac{\pi}{2}} \left(\sin 2mx\right)^{2m} dx \right)^{\frac{1}{2m}}$.

Let us evaluate the integral of the general form on the right-hand side: $\left(\int_0^{\frac{\pi}{2}} \left(\sin kx\right)^{2m} dx \right)^{\frac{1}{2m}}, \ k=1,\ldots,2m$.

Let's make a substitution in it: $t=kx : 0 \to \frac{\pi}{2}k \ \forall x : 0 \to \frac{\pi}{2}, dx = \frac{1}{k}dt$ and as a result we get:

$$\left(\int_0^{\frac{\pi}{2}} \left(\sin kx\right)^{2m} dx \right)^{\frac{1}{2m}} = \left(\frac{1}{k} \int_0^{\frac{\pi}{2}k} \left(\sin t\right)^{2m} dt \right)^{\frac{1}{2m}} = \left(\frac{k}{k} \int_0^{\frac{\pi}{2}} \left(\sin t\right)^{2m} dt \right)^{\frac{1}{2m}} = \left(\int_0^{\frac{\pi}{2}} \left(\sin t\right)^{2m} dt \right)^{\frac{1}{2m}} = \left(\frac{1}{2} \frac{\Gamma\left(\frac{2m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+1\right)} \right)^{\frac{1}{2m}} =$$

$$= \left(\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(m\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m\right)m!} \right)^{\frac{1}{2m}} = \left(\frac{\sqrt{\pi}}{2} \frac{\sqrt{\pi}}{2^{2m-1}} \frac{\Gamma\left(2m\right)}{m\left(\left(m-1\right)!\right)^2} \right)^{\frac{1}{2m}} = \left(\frac{\pi}{2^{2m}} \frac{\left(2m-1\right)!}{m\left(\left(m-1\right)!\right)^2} \right)^{\frac{1}{2m}} = \frac{1}{2} \left(\frac{\pi\left(2m-1\right)!}{m\left(\left(m-1\right)!\right)^2} \right)^{\frac{1}{2m}} .$$

Note that Legendre's formula for Euler's gamma function was used here: $\Gamma\left(a\right)\Gamma\left(a+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma\left(2a\right), a>0$. Hence

$$\int_0^{\frac{\pi}{2}} \left|\sin x \sin 2x \ldots \sin 2mx\right| dx \le \frac{\pi}{2^{2m}} \frac{\left(2m-1\right)!}{m\left(\left(m-1\right)!\right)^2} .$$

CONCLUSIONS AND PROSPECTS FOR FURTHER RESEARCH

To obtain further results, to prove new meaningful inequalities, it is necessary to make a successful selection of vectors or functions and apply to them the generalized inequalities of Hölder, Cauchy-Buniakovsky or Minkowski. One can also set the task of giving an interpretation of the obtained results from the point of view of the theory of polylinear forms. It is also interesting to prove similar inequalities for multiple integrals of a function of many variables.

CONFLICT OF INTEREST

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DATA AVAILABILITY

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USE OF ARTIFICIAL INTELLIGENCE (AI) TOOLS

AI tools were not used in the writing of this work.

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