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Conditions of Dedekindness of generalized norms in nonperiodic groups

T. D. Lukashova*, M. G. Drushlyak† and F. M. Lyman‡

Department of Mathematics
Makarenko Sumy State Pedagogical University
Romenska, 87, Sumy, Ukraine
**tanya.lukashova2015@gmail.com*
†marydru@fizmatsspu.sumy.ua
‡mathematicSSPU@gmail.com

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The authors consider generalized norms for different systems of infinite and noncyclic subgroups in nonperiodic groups. Relations between these norms are established. The conditions under which the given norms are Dedekind, in particular, central, are studied.

Keywords: Norm of group; generalized norm; norm of infinite subgroups; norm of infinite cyclic subgroups; norm of cyclic subgroups of nonprime order; norm of infinite Abelian subgroups; norm of noncyclic subgroups; norm of Abelian noncyclic subgroups.

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1. Introduction

In group theory, findings related to the study of characteristic subgroups (in particular, the center, the derived subgroup, Frattini subgroup, etc.) and the impact of properties of these subgroups on the structure of a group are in the focus. Nowadays the list of such characteristic subgroups can be broadened by means of different Σ-norms of a group.

Let Σ be the system of all subgroups of a group which have some theoretical group property. The intersection $N_\Sigma(G)$ of the normalizers of all subgroups of a group G which belong to the system Σ is called Σ-norm of a group G. In the case $\Sigma = \emptyset$, we assume that $G = N_\Sigma(G)$.

For the first time, the notion of the Σ-norm was introduced by Baer in 1934 for the system Σ of all subgroups of a group G. Such a Σ-norm was called the norm $N(G)$ of a group G and denoted as the intersection of normalizers of all subgroups of a group. Later, the properties of the norm of a group and its impact on the properties of a group were studied by Baer and his followers.

It should be noted that this direction of research has been quite widespread not only in group theory, as evidenced by the findings related to generalized norms of groups for systems of subgroups with different theoretical group properties [1, 6, 7, 10, 14–17], but also in ring theory while investigating Baer-kernel, which is a ring analogue of the norm of a group [3], and in vector space theory where the norm of vector subspace is considered in linear groups with finite dimensional orbits [5].

In the study of groups with generalized norms, a number of directions can be identified:

— the study of groups which coincide with their Σ-norms, i.e. groups, in which all subgroups of the system Σ are normal [4, 8, 9, 11–13];
— the study of groups in which Σ-norms degenerate into a unit subgroup (or the center [2]);
— the study of groups which have noncentral Dedekind Σ-norms;
— the study of groups which have proper non-Dedekind Σ-norms [13–15];
— the study of infinite groups which have Σ-norms of finite nonindentity index [10].

Among these directions the authors are most interested in the study of groups with non-Dedekind Σ-norms. But in the course of conducting such studies, it turns out that the findings concerning the diametrically opposite case, that is, the conditions under which the Σ-norm is Dedekind, are also useful. Recall that a group is called Dedekind if all its subgroups are normal. A Σ-norm is called Dedekind if all subgroups are normal in it.

In this paper, the authors consider the conditions of Dedekindness for the following Σ-norms:

— the norm $N_G(\infty)$ of infinite subgroups;
— the norm $N_G(A_\infty)$ of infinite Abelian subgroups;
— the norm $N_G(C_\infty)$ of infinite cyclic subgroups;
— the norm $N_G(C_{\overline{p}})$ of cyclic subgroups of nonprime order (in particular, infinite order);
— the norm N_G of noncyclic subgroups;
— the norm N_G^A of Abelian noncyclic subgroups.

In the next E is the trivial subgroup of a group G and $\leftthreetimes$ means the symbol of a semidirect product.

Note that non-Abelian nonperiodic groups which coincide with given Σ-norms and contain at least one Σ subgroup are described in [4, 8, 9, 11–13].

2. Preliminary Results

Lemma 2.1. *Let Σ be a system of subgroups of a group G and for every Σ-subgroup S the subgroup $S \times \langle x \rangle$, where $x \in G$, is also a Σ-subgroup. If a group G contains* *a Σ-subgroup A, which has identity intersection with Σ-norm of a group G, then Σ-norm is Dedekind.*

Proof. Let $N_G(\Sigma)$ be a Σ-norm of a group G and A be a Σ-subgroup such that $A \cap N_G(\Sigma) = E$. Then $[A, \langle x\rangle] \subseteq A \cap N_G(\Sigma) = E$ for an arbitrary element $x \in N_G(\Sigma)$. By the condition of the lemma $\langle A, x\rangle = A \times \langle x\rangle$ is also a Σ-subgroup and normalized by $N_G(\Sigma)$. But in this case

$$(A \times \langle x\rangle) \cap N_G(\Sigma) = \langle x\rangle \lhd N_G(\Sigma).$$

Thus, the norm $N_G(\Sigma)$ is Dedekind. The lemma is proved. □

Corollary 2.1. *In a nonperiodic group G with the non-Dedekind Σ-norm, where Σ is the system of either all infinite, or all infinite Abelian, or all noncyclic, or all Abelian noncyclic subgroups of a group, an arbitrary Σ-subgroup has a nonidentity intersection with the Σ-norm.*

The following statement describes the relations between different Σ-norms in nonperiodic groups.

Lemma 2.2. *In a nonperiodic group G the following relations take place*

$$Z(G) \subseteq N(G) \subseteq N_G(\infty) \subseteq N_G(A_\infty) \subseteq N_G(C_\infty),$$
$$Z(G) \subseteq N(G) \subseteq N_G(C_{\overline{p}}) \subseteq N_G(C_\infty),$$
$$Z(G) \subseteq N(G) \subseteq N_G \subseteq N_G^A.$$

The proof of the lemma follows from the definitions of the corresponding Σ-norm.

By Lemma 2.2, the Dedekindness of the norms $N_G(\infty)$, $N_G(A_\infty)$, $N_G(C_{\overline{p}})$ follows from the Dedekindness of the norm $N_G(C_\infty)$ of infinite cyclic subgroups. Therefore, we will further investigate the conditions under which the norm $N_G(C_\infty)$ is Dedekind.

Note that all infinite cyclic subgroups of a group G are normal in the case $G = N_G(C_\infty)$. So by the results of [13] a group G is either Abelian or a group of the type

$$G = A\langle b\rangle,$$

where A is a nonperiodic Abelian group, $b^4 = 1$ and $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$.

Thus in torsion-free groups the norm $N_G(C_\infty)$ is Abelian, so are $N_G(A_\infty)$, $N_G(C_\infty)$, $N_G(C_{\overline{p}})$ are Abelian. Moreover, in this case, the following statement takes place.

Lemma 2.3. *If G is a torsion-free group, then its norm $N_G(C_\infty)$ of infinite cyclic subgroups coincides with the center $Z(G)$ of a group.*

Proof. Let G be a torsion-free group and $N_G(C_\infty)$ be the norm of infinite cyclic subgroups of a group G. Then $N_G(C_\infty)$ is Abelian by [13].

Suppose that $N_G(C_\infty) \neq Z(G)$. Then elements $x \in N_G(C_\infty)$, $y \in G$ with $[x, y] \neq 1$ exist. Since the group G is torsion-free, $|y| = \infty$ and the subgroup $\langle y \rangle$ is $N_G(C_\infty)$-admissible. Therefore $x^{-1}yx = y^{-1}$ and $\langle x \rangle \cap \langle y \rangle = E$. Since $[x^2, y] = 1$, $\langle x^2 y \rangle$ is x-admissible subgroup and $x^{-1}x^2yx = x^{-2}y^{-1} = x^2y^{-1}$. But in this case $x^4 = 1$, which contradicts the condition. Thus $N_G(C_\infty) = Z(G)$. The lemma is proved. $\square$

The following statement is a direct consequence of Lemmas 2.2 and 2.3.

Corollary 2.2. *If G is a torsion-free group, then*

$$N_G(C_\infty) = N_G(A_\infty) = N_G(\infty) = N_G(C_{\overline{p}}) = N(G) = Z(G).$$

Corollary 2.3. *An arbitrary torsion-free group, which is a finite extension of the Σ-norm, where Σ is the system of either all infinite, or all infinite Abelian, or all infinite cyclic, all cyclic subgroups of nonprime order, is Abelian.*

Proof. By Lemmas 2.3 and 2.4, $N_G(\Sigma) = Z(G)$ for every above-mentioned system Σ and, hence, $[G : Z(G)] < \infty$. Therefore $|G'| < \infty$. Since G is a torsion-free group, $G' = E$ and G is Abelian. $\square$

Lemma 2.4. *If the center $Z(G)$ of a nonperiodic group G contains elements of infinite order, then the norm $N_G(C_\infty)$ coincides with the center $Z(G)$.*

Proof. By the description of groups, all infinite cyclic subgroups of whose are normal [13], and by condition the norm $N_G(C_\infty)$ is Abelian. Let us show that in this case every element from $N_G(C_\infty)$ is permutable with all elements of infinite order of a group G.

Let $x \in N_G(C_\infty)$, $y \in G$, $|y| = \infty$ and $[x, y] \neq 1$. Since $N_G(C_\infty)$ is an Abelian nonperiodic group, it is generated by elements of infinite order. So, we can consider that $|x| = \infty$. Then $x^{-1}yx = y^{-1}$ and $\langle x \rangle \cap \langle y \rangle = E$. On account of $[x^2, y] = 1$ and the subgroup $\langle x^2 y \rangle$ is x-admissible, $x^{-1}x^2yx = x^{-2}y^{-1} = x^2y^{-1}$. Therefore $x^4 = 1$, contrary to its choice. Thus, $[x, y] = 1$ for arbitrary elements $x \in N_G(C_\infty)$ and $y \in G$, $|y| = \infty$.

Let $y \in G$, $|y| < \infty$. Then $|yz| = \infty$, where $z \in Z(G)$, $|z| = \infty$. In the same manner, we can see that $[\langle y \rangle, N_G(C_\infty)] = E$. Thus $N_G(C_\infty) = Z(G)$. The lemma is proved. $\square$

Corollary 2.4. *If the center $Z(G)$ of a group G contains elements of infinite order, then*

$$N_G(C_\infty) = N_G(A_\infty) = N_G(\infty) = N_G(C_{\overline{p}}) = N(G) = Z(G).$$

3. Main Results

Let us consider the conditions under which the mentioned generalized norms are Dedekind. The following statement generalizes Corollary 2.2 and characterizes the norm of infinite subgroups in nonperiodic groups.

Theorem 3.1. *In a nonperiodic group G the norm $N_G(\infty)$ of infinite subgroups is Abelian. Moreover, $N_G(\infty)$ coincides with the center of a group G, if it is a nonperiodic group.*

Proof. Let the norm $N_G(\infty)$ be torsion and $x \in G$ be an arbitrary element of infinite order. Since the subgroup $\langle x \rangle$ is $N_G(\infty)$-admissible, it is normal in the group $G_1 = \langle x \rangle N_G(\infty)$. Therefore $G_1 = \langle x \rangle \times N_G(\infty)$ and $x \in Z(G_1)$. By Corollary 2.4 $N_{G_1}(\infty) = Z(G_1)$. Thus, the norm $N_{G_1}(\infty)$ of the subgroup G_1 is Abelian and by $N_G(\infty) \subseteq N_{G_1}(\infty)$ the norm $N_G(\infty)$ is also Abelian.

Let us consider the case when the norm $N_G(\infty)$ is nonperiodic. By Lemma 2.2 $N_G(\infty) \subseteq N_G(C_\infty)$. Thus the norm $N_G(C_\infty)$ is also nonperiodic. On account of the description of nonperiodic groups, in which all infinite cyclic subgroups are normal [13], the group $N_G(C_\infty)$ and with it the group $N_G(\infty)$ are soluble. Therefore $N_G(\infty)$ is Abelian by [4]. Let us show that in this case $N_G(\infty) = Z(G)$.

Suppose that elements $a \in N_G(\infty)$ and $x \in G$ with $[a, x] \neq 1$ exist. If $|x| = \infty$, then the subgroup $\langle x \rangle$ is $N_G(\infty)$-admissible. Then by $[a, x] \neq 1$, we conclude that $N_G(\infty) \cap \langle x \rangle \neq E$. Thus $x^k \in N_G(\infty)$ for some natural number k. Then $a^{-1}x^k a = x^{-k} = x^k$ and $x^{2k} = 1$, contrary to the choice of the element x. From this, we conclude that $N_G(\infty) \subseteq C_G(\langle x \rangle)$ for any element x of infinite order of a group G.

Let $|x| < \infty$, $|a| = \infty$. Let us consider a normal closure $A = \langle a \rangle^{\langle x \rangle}$ of the subgroup $\langle a \rangle$ in the group $G_1 = \langle x \rangle N_G(\infty)$. The subgroup A is finitely generated nonperiodic Abelian. Its torsion part $T(A)$ is finite, so $A^{|T(A)|} = A_1$ is an Abelian torsion-free group of finite rank and $A_1 \lhd G_1$. Then $A_1^n \lhd G_1$ for any natural number n. Thus,

$$\bigcap_{n=1}^{\infty} A_1^n \langle x \rangle = \langle x \rangle \lhd G_1$$

and, hence, $[A_1, \langle x \rangle] = E$. Let $a_1 \in A_1$ and $|a_1| = \infty$. Then $|a_1 x| = \infty$ and by the proved above $N_G(\infty) \subseteq C_G(\langle a_1 x \rangle)$. Therefore $N_G(\infty) \subseteq C_G(\langle x \rangle)$, i.e. $N_G(\infty) \subseteq Z(G)$ and, hence, $N_G(\infty) = Z(G)$. The theorem is proved. □

The following example confirms that in Theorem 3.1 in proving the equality $N_G(\infty) = Z(G)$, we cannot ignore the nonperiodicness of the norm $N_G(\infty)$.

Example 3.1. In the group $G = (\langle a \rangle \times \langle b \rangle)\langle c \rangle$, where $|a| = |c| = 4$, $|b| = \infty$, $c^2 = a^2$, $c^{-1}ac = a^{-1}$, $c^{-1}bc = b^{-1}$, the norm $N_G(\infty) = \langle a \rangle$ of infinite subgroups is an Abelian torsion group which is different from the center $Z(G) = \langle a^2 \rangle$ of the group.

Indeed, $N_G(\infty) \subseteq N_G(\langle b^n, c\rangle \cap \langle b^4, bc\rangle) = \langle a\rangle$, where n is natural number, the element a is contained in the normalizer of every infinite subgroup.

Note that all infinite cyclic and all infinite Abelian subgroups are normal in this group. Therefore $N_G(C_\infty) = N_G(A_\infty) = G$.

Corollary 3.1. *If the norm $N_G(\infty)$ of a nonperiodic group G has a finite index in a group, then $N_G(\infty) = Z(G)$.*

Summarizing the above results, we give sufficient conditions of the Dedekindness of Σ-norms for systems of infinite Abelian, infinite cyclic and cyclic subgroups of non-prime order.

Theorem 3.2. *In a nonperiodic group G the norms $N_G(C_\infty)$, $N_G(A_\infty)$, $N_G(C_{\overline{p}})$ are Abelian in each of the following cases*:

(1) *the center $Z(G)$ contains elements of infinite order*;
(2) *any of the given Σ-norms is torsion*;
(3) *a group G contains an infinite cyclic subgroup $\langle x\rangle$ which has an identity intersection with Σ-norm $(N_G(C_\infty)$, $N_G(A_\infty)$, $N_G(C_{\overline{p}}))$*;
(4) *a group G contains a Σ-subgroup A which has an identity intersection with Σ-norm*;
(5) *G is an involution-free group.*

Proof. (1) The first statement of the theorem follows from Corollary 2.4.

(2) Let the norm $N_G(\Sigma)$ for any of mentioned systems of subgroups be torsion. Then $\langle x\rangle \cap N_G(\Sigma) = E$ for an arbitrary element $x \in G$, $|x| = \infty$. Since the subgroups $\langle x\rangle$ and $N_G(\Sigma)$ are normal in the group $G_1 = \langle x\rangle N_G(\Sigma)$, so $G_1 = N_G(\Sigma) \times \langle x\rangle$. Thus, $x \in Z(G_1)$ and by the proved above both the norm $N_{G_1}(\Sigma)$ and the norm $N_G(\Sigma)$ are Abelian.

(3) Let $x \in G$, $|x| = \infty$ and $\langle x\rangle \bigcap N_G(\Sigma) = E$, where $N_G(\Sigma)$ be the norm of infinite cyclic, infinite Abelian or cyclic subgroups of nonprime order. Since $\langle x\rangle$ is $N_G(\Sigma)$-admissible, $x \in Z(G_1)$, where $G_1 = \langle x\rangle N_G(\Sigma)$. By the proved in the case (1) the norm $N_{G_1}(\Sigma)$ is Abelian, hence, the norm $N_G(\Sigma) \subseteq N_{G_1}(\Sigma)$ is also Abelian.

(4) If Σ consists of all infinite Abelian or all infinite cyclic subgroups, then the Σ-norms are Dedekind by Lemma 2.1 and the above case of the theorem, respectively.

Let Σ be the system of all cyclic subgroups of nonprime order and $A = \langle x\rangle$ be a Σ-subgroup such that $\langle x\rangle \cap N_G(C_{\overline{p}}) = E$. If $|x| = \infty$ or the norm $N_G(C_{\overline{p}})$ is torsion, then it is Abelian by the proved above. Thus, it remains to exclude the case, when $N_G(C_{\overline{p}})$ is a nonperiodic group and the subgroup $\langle x\rangle$ has nonprime order.

Suppose that the subgroup $N_G(C_{\overline{p}})$ is non-Abelian. Then $N_G(C_{\overline{p}})$ is a nonperiodic almost Dedekind group. By [8] $N_G(C_{\overline{p}}) = C \rtimes \langle b\rangle$, where C is a nonperiodic Abelian group, $|b| = |bc| = 2$, $b^{-1}cb = c^{-1}$ for any element $c \in C$. Since the subgroup $\langle x\rangle$ is $N_G(C_{\overline{p}})$-admissible, $\langle x\rangle \subseteq Z(G_1)$, where $G_1 = \langle x\rangle N_G(C_{\overline{p}})$. Thus, G_1

is an almost Dedekind group, whose center contains elements of composite order, which is impossible. So the norm $N_G(C_{\overline{p}})$ is Abelian.

(5) If G is an involution-free group. Then its norm $N_G(C_\infty)$ of infinite cyclic subgroups is Abelian. Thus, all mentioned generalized norms are Abelian.

The theorem is proved. □

Corollary 3.2. *If the center $Z(G)$ of a nonperiodic group G contains elements of composite order, then the norm $N_G(C_{\overline{p}})$ is Abelian.*

Proof. On account of the above theorem, it is enough to consider the case when the norm $N_G(C_{\overline{p}})$ is nonperiodic.

Suppose that the subgroup $N_G(C_{\overline{p}})$ is non-Abelian. So it is a nonperiodic almost Dedekind group and by [8] $N_G(C_{\overline{p}}) = C \leftthreetimes \langle b\rangle$, where C is a nonperiodic Abelian group, $|b| = |bc| = 2$, $b^{-1}cb = c^{-1}$ for any element $c \in C$. Since the center of such a group does not contain elements of composite order, the assumption is false and the norm $N_G(C_{\overline{p}})$ is Abelian. □

Corollary 3.3. *In a nonperiodic group G the norms of infinite cyclic, infinite Abelian and infinite subgroups of nonprime order are either Abelian (torsion or nonperiodic) or non-Abelian nonperiodic groups whose corresponding systems of infinite subgroups are normal.*

Thus, the condition of Dedekindness of the norms of infinite, infinite cyclic, infinite Abelian and infinite subgroups of nonprime order of nonperiodic groups is equivalent to the condition of Abelity of these norms.

The task is now to find the conditions under which the norm of noncyclic and the norm of Abelian noncyclic subgroups are Dedekind. Note that non-Abelian groups, whose all noncyclic or Abelian noncyclic subgroups (provided such subgroups exist in a group) are normal, were studied in [9, 11, 12] and were called $\overline{H}$-groups and $\overline{HA}$-groups, respectively.

Theorem 3.3. *The noncyclic norm N_G of a nonperiodic locally soluble by finite group G is Dedekind, if the one of the following conditions takes place:*

(1) *a group G contains a noncyclic subgroup A such that $A \cap N_G = E$;*
(2) *the noncyclic norm N_G of a group G is torsion;*
(3) *a group G contains a nonidentity cyclic N_G-admissible subgroup $\langle g\rangle$, such that $\langle g\rangle \cap N_G = E$;*
(4) *a group G contains a free Abelian subgroup of rank* 2;
(5) *a group G has the torsion center $Z(G)$;*
(6) *a group G contains a finite Abelian noncyclic subgroup;*
(7) *G is a torsion-free group.*

Proof. (1) The validity of this assertion follows from Lemma 2.1 for the system Σ of all noncyclic subgroups of a group.

(2) Let N_G be a torsion non-Dedekind group. Let us consider an element $x \in G$, such that $|x| = \infty$ and the group $G_1 = \langle x \rangle N_G$. By the description of torsion non-Hamiltonian $\overline{H}$-groups (see [11, 12]), to which the norm N_G applies, this norm contains a finite characteristic subgroup M. Then $M \lhd G_1$, $C_{G_1}(M) \lhd G_1$ and $[G_1 : C_{G_1}(M)] < \infty$. Thus, the element $x_1 \in \langle x \rangle$, $|x_1| = \infty$, $[\langle x_1 \rangle, M] = 1$ exists. But then $\langle x_1, M \rangle$ is noncyclic, so it is N_G-admissible.

Let $|M| = m$. Then the subgroup $\langle x_1^m \rangle$ is also N_G-admissible. Since $[\langle x_1^m \rangle, N_G] \subseteq \langle x_1^m \rangle \cap N_G = E$, the subgroup it is $\langle a, x_1^m \rangle$ is Abelian noncyclic for an arbitrary element $a \in N_G$ and, hence, i N_G-admissible. But then

$$\langle a \rangle = \bigcap_{k=1}^{\infty} \langle a, x_1^{km} \rangle \cap N_G \lhd N_G.$$

for any natural number k

Therefore N_G is Dedekind.

(3) Let $\langle g \rangle$ be a cyclic subgroup which satisfies the condition (3) of the theorem. Suppose that the norm N_G is non-Dedekind. Then it is nonperiodic and a non-Hamiltonian $\overline{H}$-group of one of the types:

(i) $G = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = p^n$, $n \neq 1$ ($n > 1$ if $p = 2$), $|b| = \infty$, $[a, b] = a^{p^{n-1}}$;
(ii) $G = H \times B$, where $H \leq h_1, h_2 >$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$, B is an infinite cyclic group or a group isomorphic to an additive group of dyadic numbers.

Since

$$[\langle g \rangle, N_G] \subseteq \langle g \rangle \cap N_G = E,$$

the subgroup $\langle g, x \rangle$ is Abelian noncyclic for an arbitrary element $x \in N_G$, $|x| = \infty$. Therefore,

$$\langle g, x \rangle \cap N_G = \langle x \rangle \lhd N_G$$

and all infinite cyclic subgroups are normal in the norm N_G, which contradicts the properties of the subgroup N_G.

(4) Let a group G contain a free Abelian subgroup $A = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = \infty$, and its noncyclic norm N_G be non-Dedekind. By the proved above N_G is a nonperiodic $\overline{H}$-group of one of the types mentioned in (3).

Since the norm N_G does not contain free Abelian subgroups of rank 2, we can assume that $\langle x \rangle \cap N_G = E$. Taking into account that the subgroup $\langle x, y^k \rangle$ is N_G-admissible for any natural number k, the subgroup $\langle x \rangle = \bigcap_{k=1}^{\infty} \langle x, y^k \rangle$ is also N_G-admissible. By the condition (3) of the theorem the subgroup N_G is Dedekind.

(5) Suppose that the norm N_G of noncyclic subgroups of a group G is non-Dedekind. Then by the proved above N_G is a nonperiodic $\overline{H}$-group of one of two types mentioned in (3).

Let denote by Z a natural power of the center $Z(N_G)$ which does not contain nonidentity elements of finite order. Then Z is an infinite cyclic group or a group isomorphic to an additive group of dyadic numbers. We are now in a position to show that $Z \subseteq Z(G)$.

Let $x \in G$, $|x| < \infty$. Then $\langle x, Z^{p^k}\rangle \lhd G_1 = \langle x\rangle N_G$ for a prime number $p \neq 2$ and an arbitrary natural number k. Therefore

$$\bigcap_{k=1}^{\infty} \langle Z^{p^k}, x\rangle = \langle x\rangle \lhd G_1$$

and $[\langle x\rangle, Z] \subseteq Z \cap \langle x\rangle = E$.

Let us prove that the subgroup Z centralizers every element $x \in G$ of infinite order. Let $z \in Z$ and $x^{-1}zx = z_1$, $z_1 \in Z$. Since $\langle x\rangle \cap Z \neq E$, $z^n \in \langle x\rangle$ for some natural number n by the locally cyclicity of the group Z. So $x^{-1}z^n x = z_1^n = z^n$, $z = z_1$ and $[x, z] = 1$. Thus, $Z \subseteq Z(G)$ and the center $Z(G)$ of a group contains elements of infinite order, which contradicts the condition. So, the noncyclic norm of a group can not be non-Dedekind.

(6) Let A be a finite Abelian noncyclic subgroup of a group G. Suppose, contrary to the condition, that the noncyclic norm N_G is non-Dedekind. Since the center $Z(G)$ is nonperiodic by the proved above, the subgroup $\langle a, z\rangle$ is noncyclic for arbitrary elements $a \in A$, $a \neq 1$ and $z \in Z(G)$, $|z| = \infty$, and therefore it is N_G-admissible. So the subgroup $\langle a\rangle$ is also N_G-admissible. On account of $A \nsubseteq N_G$, we can point such an element $x \in A$, $\langle x\rangle \cap N_G = E$. Since the subgroup $\langle x\rangle$ is N_G-admissible, the noncyclic norm N_G is Dedekind by (3) of the theorem. The contradiction.

(7) Let G be a torsion-free group. Suppose that its noncyclic norm N_G is non-Dedekind. Then N_G is a torsion-free $\overline{H}$-group, contrary to the description of such groups [11].

The theorem is proved. □

Corollary 3.4. *If the norm N_G of noncyclic subgroups of a nonperiodic locally soluble by finite group G is non-Dedekind, then every noncyclic subgroup and every cyclic subgroup, normal in G, have a nonidentity intersection with the norm N_G.*

Corollary 3.5. *If the norm N_G of noncyclic subgroups of a nonperiodic locally soluble by finite group G is non-Dedekind, then the center $Z(G)$ of a group contains elements of infinite order.*

Corollary 3.6. *A nonperiodic locally soluble by finite group G with non-Dedekind noncyclic norm N_G does not contain finite Abelian noncyclic subgroups.*

The next task is to find conditions under which the norm N_G^A of noncyclic subgroups of a nonperiodic group is Dedekind.

Lemma 3.1. *Let G be a nonperiodic group, N_G^A be the norm of noncyclic subgroups and a group G contain a nonidentity N_G^A-admissible subgroup H such that $N_G^A \cap H = E$. If the norm N_G^A is nonperiodic, then all infinite cyclic subgroups are normal in it. In particular, N_G^A is Dedekind, if for an arbitrary nonidentity element $y \in N_G^A$ an element $h \in H$ such, that the subgroup $\langle y, h\rangle$ is noncyclic, exists.*

Proof. Since H is N_G^A-admissible and $N_G^A \lhd G$, $HN_G^A = H \times N_G^A$. Let $x \in N_G^A$ and $|x| = \infty$. Then for $h \neq 1$ and $h \in H$ the subgroup $\langle x, h\rangle$ is Abelian noncyclic and therefore N_G^A-admissible. So $\langle x, h\rangle \cap N_G^A = \langle x\rangle \lhd N_G^A$, which is the desired conclusion. Suppose that for an arbitrary element $y \in N_G^A$, $y \neq 1$ the element $h \in H$ such, that the subgroup $\langle y, h\rangle$ is noncyclic, exists. Then $\langle y, h\rangle \cap N_G^A = \langle y\rangle \lhd N_G^A$ and the subgroup N_G^A is Dedekind. The lemma is proved. □

Theorem 3.4. *The norm N_G^A of Abelian noncyclic subgroups of a nonperiodic group G is Dedekind in each of the following cases*:

(1) *a group G contains an Abelian noncyclic subgroup A such that $A \cap N_G^A = E$;*
(2) *a group G contains an infinite cyclic N_G^A-admissible subgroup $\langle g\rangle$ such that $\langle g\rangle \cap N_G^A = E$;*
(3) *the norm N_G^A is finite.*

Proof. (1)–(2) The proof of (1) and (2) of the theorem follows from Lemma 3.1.

(3) Let us show that when the condition (3) is fulfilled the norm of Abelian noncyclic subgroups is Dedekind.

Let $|N_G^A| < \infty$. Since $N_G^A \lhd G$, $[G : C_G(N_G^A)] < \infty$ and the centralizer $C_G(N_G^A)$ contains an element g of infinite order. Since the subgroup $\langle g\rangle$ is N_G^A-admissible, the further proof is reduced to the application of the case (2) of the theorem. The theorem is proved. □

We note, that under the conditions stated in Theorem 3.4, since $N_G \subseteq N_G^A$, the norm N_G of Abelian noncyclic subgroups is Dedekind.

The below example confirms that the norm of Abelian noncyclic subgroups and the norm of noncyclic subgroups of a nonperiodic group in contrast to the above-mentioned norms $N_G(C_\infty)$, $N_G(A_\infty)$, $N_G(\infty)$, $N_G(C_{\overline{p}})$ of different systems of infinite subgroups can be Hamiltonian.

Example 3.2. In the group

$$G = H \times (\langle a\rangle \leftthreetimes \langle b\rangle),$$

where $H = \langle h_1, h_2\rangle$ is a quaternion group of order 8, $|a| = \infty$, $|b| = 2$, $b^{-1}ab = a^{-1}$, both the norm N_G^A of Abelian noncyclic subgroups and the norm N_G of noncyclic subgroups are Hamiltonian and coincide with the subgroup H.

Indeed,

$$N_G \subseteq N_G^A \subseteq N_G(\langle h_1^2\rangle \times \langle ab\rangle) \cap (\langle h_1^2\rangle \times \langle b\rangle) = H.$$

Besides, every Abelian noncyclic subgroup of the group G contains $H^2 = \langle h_1^2\rangle$ and $[H, G] = \langle h_1^2\rangle$. Therefore $N_G^A = H$. In the same manner, it is easy to see that the subgroup H normalizers every noncyclic subgroup of the group G. Thus $N_G = H$.

Corollary 3.7. *If the norm N_G^A of Abelian noncyclic subgroups of a group G is non-Dedekind, then every Abelian noncyclic subgroup and every normal infinite cyclic subgroup have with the norm N_G^A a nonidentity intersection.*

Corollary 3.8. *If the norm N_G^A of Abelian noncyclic subgroups of a nonperiodic group G is non-Dedekind and torsion, then all Abelian torsion-free subgroups of a group are cyclic and N_G^A is infinite.*

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