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ON THE NORM OF DECOMPOSABLE SUBGROUPS IN NONPERIODIC GROUPS

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We study the relationships between the properties of nonperiodic groups and the norms of their decomposable subgroups. In particular, we analyze the influence of restrictions imposed on the norm of decomposable subgroups on the properties of the group in the case where this norm is non-Dedekind. We also describe the structure of nonperiodic locally nilpotent groups for which the indicated norm is non-Dedekind. Moreover, some relations between the norm of noncyclic Abelian subgroups and the norm of decomposable subgroups are established.

Introduction

Let Σ be the system of all subgroups of a group characterized by a certain group-theoretic property. Recall that the Σ-norm of the group G is defined as the intersection of normalizers of all subgroups of the group G contained in the system Σ. In particular, if Σ consists of all subgroups of the groups G, then the corresponding Σ-norm is called a norm of the group [1].

We continue our investigation of the relationships between the properties of a group and the properties of its Σ-norms for different systems of subgroups Σ. In the present paper, we consider the properties of the norms of decomposable subgroups in nonperiodic locally solvable groups and the influence of these norms on the properties of the group itself. Note that similar investigations in the class of locally finite groups were performed in [2]. Some results of the present paper were announced in [3].

Recall that a subgroup of the group G representable in the form of the direct product of two nontrivial factors is called *decomposable* [4]. Thus, the intersection of normalizers of all decomposable subgroups of the group G is called the norm of decomposable subgroups and denoted by N_G^d. For groups without decomposable subgroups, we assume that $G = N_G^d$.

It follows from the definition of the norm N_G^d that, in the case where $N_G^d = G$, the decomposable subgroups of the group G are either normal or do not exist. The non-Abelian groups with this property were studied in [4], where they were called di-groups. In what follows, we use the following two assertions that describe the structure of di-groups:

Proposition 1 [4]**.** *The family of non-Abelian locally finite and nonperiodic locally solvable di-groups all subgroups of which are indecomposable is exhausted by groups of the following types:*

(1) a (finite or infinite) quaternion 2-group;

(2) a Frobenius group $G = A \leftthreetimes B$, where A is a locally cyclic p-group, B is a cyclic q-group, p and q are prime numbers, and $(p-1, q) = q$;

(3) a Frobenius group $G = A \leftthreetimes B$, where A is a torsion-free Abelian group of rank 1 and B is an infinite cyclic group or a group of order 2.

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The semidirect product $G = A \leftthreetimes B$ of two nontrivial groups A and $B,$ where

$$B \cap g^{-1}Bg = E \quad \text{for any element} \quad g \in G \backslash B$$

and

$$A \backslash E = G \backslash \bigcup_{g \in G} \left(g^{-1}Bg\right),$$

is called a *Frobenius group* (see [5]).

Proposition 2 [4]. *The nonperiodic di-groups each of which has a decomposable subgroup are groups of one of the following types:*

(1) $G = A \leftthreetimes \langle b \rangle,$ where A is an involution-free nonperiodic Abelian group containing a decomposable subgroup, $|b| = 2,$ and $b^{-1}ab = a^{-1}$ for any element $a \in A$;

(2) $G = A\langle b \rangle,$ where A is a nonperiodic Abelian group with unique involution b^2 and $b^{-1}ab = a^{-1}$ for any element $a \in A$;

(3) $G = (\langle b^2 \rangle \times C)\langle b \rangle,$ where $|b| = 8,$ C is a torsion-free Abelian group of rank 1, the quotient group $G/\langle b^4 \rangle$ contains infinite Abelian subgroups, and all these subgroups are normal in $G/\langle b^4 \rangle$;

(4) $G = Q \times B,$ where Q is a quaternion group and B is a torsion-free Abelian group of rank 1;

(5) $G = \langle a \rangle \leftthreetimes B,$ $|a| = p^n,$ p is a prime number $(n > 1$ for $p = 2), B$ is a torsion-free incomplete Abelian group of rank 1, and the commutant of the group G has a prime order.

Hence, the structure of nonperiodic non-Dedekind locally solvable groups that coincide with their norm of decomposable subgroups is known. Thus, it is natural to study the properties of nonperiodic groups in which this norm is non-Dedekind and forms a proper subgroup.

1. Nonperiodic Groups with Non-Dedekind Norm of Decomposable Subgroups

In this section, we consider the relationships between the properties of nonperiodic locally solvable groups and their norms of decomposable subgroups under the additional condition that these norms are non-Dedekind.

In what follows, we often use the following assertion:

Lemma 1.1 [2]. *Let G be a group containing a nonidentity N_G^d-admissible subgroup H such that*

$$H \cap N_G^d = E,$$

where N_G^d is the norm of decomposable subgroups of the group G. Then the subgroup N_G^d is non-Dedekind.

By using Lemma 1.1, we can easily show that the norm N_G^d of decomposable subgroups of the nonperiodic group G is Dedekind if it is finite or contains a nonidentity finite characteristic subgroup. In particular, the following statement is true:

Corollary 1.1. *If the norm N_G^d of decomposable subgroups of a locally solvable nonperiodic group G satisfies the condition of minimality for Abelian subgroups, then it is Dedekind.*

Proof. By the condition, the norm N_G^d of decomposable subgroups of the group G is a locally solvable periodic group with condition of minimality for Abelian subgroups. Thus, in view of Theorem 4.3 from [6],

N_G^d contains a characteristic (and, hence, normal in G) finite Abelian subgroup A. Since $[G : C_G(A)] < \infty$, the subgroup $C_G(A)$ in nonperiodic. Therefore, there exists an element $x \in C_G(A)$ such that $|x| = \infty$. Then the subgroup $\langle A, x\rangle = A \times \langle x\rangle$ is decomposable and N_G^d-admissible. Hence, the subgroup $\langle x\rangle^{|A|}$ is also N_G^d-admissible and, in addition, $\langle x\rangle^{|A|} \cap N_G^d = E$. By virtue of Lemma 1.1, the norm N_G^d is Dedekind, Q.E.D.

The statements presented in what follows characterize the influence of the norm of decomposable subgroups on the properties of the group.

Theorem 1.1. *Let G be a nonperiodic group with non-Dedekind norm N_G^d of decomposable subgroups. Then any decomposable Abelian subgroup of the group G is mixed if and only if every decomposable Abelian subgroup of the norm N_G^d is mixed.*

Proof. The direct assertion of the theorem is evident. We now prove the converse assertion. Assume that all decomposable Abelian subgroups of the norm N_G^d are mixed and the group G itself contains a decomposable subgroup $M = \langle x\rangle \times \langle y\rangle$, where $|x| = |y| = \infty$ or $|x| = p,\ |y| = q$, and p and q are prime numbers. It follows from the description of nonperiodic di-groups (Proposition 2) that, in this case, N_G^d is a group of one of the following types:

(1) $N_G^d = A \leftthreetimes \langle b\rangle$, where $A = A_1 \times C$ is an involution-free nonperiodic Abelian group, A_1 is a torsion-free Abelian group of rank 1, C is a locally cyclic p-group (p is an odd prime number), $|b| = 2$, and $b^{-1}ab = a^{-1}$ for any element $a \in A$;

(2) $N_G^d = A\langle b\rangle$, where $A = A_1 \times C$ is a nonperiodic Abelian group with one involution b^2, A_1 is a torsion-free Abelian group of rank 1, C is a locally cyclic 2-group, and $b^{-1}ab = a^{-1}$ for any element $a \in A$;

(3) $N_G^d = (\langle b^2\rangle \times C)\langle b\rangle$, where $|b| = 8$, C is a torsion-free Abelian group of rank 1, and all infinite Abelian subgroups of the quotient group $G/\langle b^4\rangle$ are normal;

(4) $N_G^d = Q \times B$, where Q is a quaternion group and B is a torsion-free Abelian group of rank 1;

(5) $N_G^d = \langle a\rangle \leftthreetimes B$, where $|a| = p^n$, p is a prime number $(n > 1$ for $p = 2)$, B is a torsion-free incomplete Abelian group of rank 1, and the order of commutant of the group G is prime.

In each of these cases, the norm N_G^d contains a finite nonidentity Abelian subgroup F normal in G, whence it follows that $[G : C_G(F)] < \infty$.

If $|M| = \infty$, then, in view of the fact that all torsion-free Abelian subgroups have rank 1 in the norm N_G^d, we can assume that $N_G^d \cap \langle y\rangle = E$. Further, it follows from the condition $[G : C_G(F)] < \infty$ that $y^m \in C_M(F)$ for some $m \in N$. Hence, $\langle F, y^m\rangle$ is an N_G^d-admissible subgroup. Thus, the subgroup

$$\langle F, y^m\rangle^{|F|} = \left\langle y^{m|F|}\right\rangle$$

is also N_G^d-admissible. By using Lemma 1.1, we conclude that, in this case, contrary to the condition, the norm N_G^d is also Dedekind.

Assume that $|M| < \infty$ and $|M| = pq$. In this case, the norm N_G^d contains an element a such that $|a| = \infty$ and $a \in C_G(M)$. In this case, the subgroups $\langle a, x\rangle$ and $\langle a, y\rangle$ are N_G^d-admissible. Hence, the subgroups $\langle x\rangle$ and $\langle y\rangle$ are also N_G^d-admissible. Since $\langle x, y\rangle \nsubseteq N_G^d$, at least one of the subgroups $\langle x\rangle$ or $\langle y\rangle$ does not belong to N_G^d. By using Lemma 1.1, we conclude that, in this case, the norm N_G^d is Dedekind.

Theorem 1.1 is proved.

Corollary 1.2. *If the norm N_G^d of decomposable Abelian subgroups of the nonperiodic group G is non-Dedekind and all its decomposable Abelian subgroups are mixed, then the quotient group G/N_G^d is periodic.*

Lemma 1.2. *If a nonperiodic group G contains an Abelian subgroup M, which is either a free group of rank $r \geq 2$ or a periodic nonprimary group, then each subgroup from M is N_G^d-admissible in G.*

Proof. We restrict ourselves to the case

$$M = \langle x_1 \rangle \times \langle x_2 \rangle ,$$

where $|x_1| = |x_2| = \infty$ or $|x_1| = p^m,\ |x_2| = q^n,\ n, m \in N$, and p and q are different prime numbers.

In the first case, for any nonidentity element $x \in M$, there exists an element $y \in M$ such that $|y| = \infty$ and $\langle x \rangle \cap \langle y \rangle = E$. Then, for each natural number k, the subgroup $\left\langle x, y^k \right\rangle$ is N_G^d-admissible. Hence, the subgroup

$$\bigcap_{k=1}^{\infty} \left\langle x, y^k \right\rangle = \langle x \rangle$$

is also N_G^d-admissible. Therefore, all subgroups from M are N_G^d-admissible.

In the second case, the decomposability of M implies that it is N_G^d-admissible. Hence, its characteristic subgroups $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are N_G^d-admissible. As a result, we conclude that each subgroup from M is N_G^d-admissible.

Lemma 1.2 is proved.

Theorem 1.2. *A nonperiodic group G with non-Dedekind norm N_G^d of decomposable subgroups does not contain decomposable subgroups if and only if its norm N_G^d does not contain these subgroups.*

Proof. The direct statement of the theorem is obvious. We now prove the converse assertion. Assume that the norm N_G^d of decomposable subgroups of the group G is non-Dedekind and does not contain decomposable subgroups and that the group G contains these subgroups. Then G contains the direct product $M = \langle x \rangle \times \langle y \rangle$ of two nonidentity cyclic subgroups.

If $|x| = |y| = \infty$, then, by Lemma 1.2, each subgroup from M is N_G^d-admissible. This implies that there exists an infinite subgroup $\langle a \rangle \subset M$ such that $\langle a \rangle \bigcap N_G^d = E$. By virtue of Lemma 1.1, the norm N_G^d is Dedekind, which is impossible by the condition.

Further, we consider the case where the subgroup M is mixed and $|x| = \infty,\ |y| < \infty$. Let $y_1 \in \langle y \rangle$, $|y_1| = p$, where p is a prime number. Then the subgroups $\langle y_1 \rangle$ and $\langle x^p \rangle$ are N_G^d-admissible and at least one of them has the trivial intersection with N_G^d. By using Lemma 1.1 once again, we arrive at a contradiction.

Now let $|M| < \infty$ and $x_1 \in \langle x \rangle,\ y_1 \in \langle y \rangle$, where $|x_1| = p,\ |y_1| = q$, and p and q are prime numbers. If $p \neq q$, then the subgroups $\langle x_1 \rangle$ and $\langle y_1 \rangle$ are N_G^d-admissible and at least one of them does not belong to N_G^d. By using Lemma 1.1, we arrive at a contradiction. Hence, $p = q$ and $\langle x_1, y_1 \rangle$ is an elementary Abelian group of order p^2.

Consider a subgroup

$$G_1 = \langle x_1, y_1 \rangle\, N_G^d.$$

Since $\langle x_1, y_1 \rangle \lhd G_1$, we have

$$[G_1 : C_{G_1} \langle x_1, y_1 \rangle] < \infty.$$

If the norm N_G^d contains elements of infinite order, then $C_{G_1} \langle x_1, y_1 \rangle$ is a nonperiodic group and one can easily show that the subgroups $\langle x_1 \rangle$ and $\langle y_1 \rangle$ are N_G^d-admissible. In this case, we get the already studied case and, hence, N_G^d is a Dedekind group, which is impossible by the condition. Therefore, N_G^d is a periodic group. By virtue of Lemma 1.1, we obtain $\langle x_1, y_1 \rangle \cap N_G^d \neq E$. Then

$$\left| \langle x_1, y_1 \rangle \cap N_G^d \right| = p$$

and, without loss of generality, we can assume that $x_1 \notin N_G^d$ and $y_1 \in N_G^d$. It is clear that

$$\left[\langle x_1, y_1\rangle, N_G^d\right] = \langle y_1\rangle \lhd G_1.$$

Moreover, if the norm N_G^d is a p-group, then $\langle y_1\rangle \subseteq Z(N_G^d)$ is a unique subgroup of order p in the norm N_G^d and $\langle y_1\rangle \lhd G$. In this case, there exists an element of the group of infinite order such that $[a, y_1] = 1$. Hence, $\langle a^p\rangle$ is an N_G^d-admissible subgroup, which is impossible by virtue of Lemma 1.1. Therefore, N_G^d is a nonprimary group and there exists an element $b \in N_G^d$ such that $|b| \neq 1$ and $(|b|, p) = 1$. By the Maschke theorem,

$$\langle x_1, y_1\rangle \leftthreetimes \langle b\rangle = (\langle y_1\rangle \times \langle y_2\rangle) \leftthreetimes \langle b\rangle,$$

where $\langle y_2\rangle$ is a $\langle b\rangle$-admissible subgroup, $\langle y_2\rangle \cap N_G^d = E$, and $[y_2, b] \in N_G^d \bigcap \langle y_2\rangle = E$. Since the subgroup $\langle y_2, b\rangle$ is N_G^d-admissible, its characteristic subgroup $\langle y_2\rangle$ is also N_G^d-admissible. In this case, by virtue of Lemma 1.1, the norm N_G^d is Dedekind, which is impossible.

Theorem 1.2 is proved.

In Theorem 1.2, the condition that the norm N_G^d of decomposable subgroups is non-Dedekind is essential. The following example confirms this fact:

Example 1.1. Consider a group

$$G = (\langle a\rangle \leftthreetimes \langle b\rangle) \times \langle c\rangle,$$

where $|a| = |c| = \infty$, $|b| = 2$, and $b^{-1}ab = a^{-1}$. For this group, $N_G^d = \langle c\rangle$ and does not contain decomposable subgroups, whereas the number of these subgroups in the group G is infinite.

Corollary 1.3. *If the norm N_G^d of a nonperiodic locally solvable group G is non-Dedekind, contains elements of infinite order, and does not contain decomposable subgroups, then $G = N_G^d$ and G is a Frobenius group of type (3) in Proposition 1.*

Theorem 1.3. *A nonperiodic group G with non-Dedekind norm N_G^d of decomposable subgroups contains a free Abelian subgroup of rank $r \geq 2$ if and only if its norm N_G^d contains a subgroup of the same rank.*

Proof. If the norm N_G^d contains a free Abelian subgroup of rank $r \geq 2$, then the assertion of the theorem is obvious. Assume that G contains a free Abelian subgroup

$$M = \langle x_1\rangle \times \langle x_2\rangle \times \ldots \times \langle x_r\rangle$$

of rank $r \geq 2$. By virtue of Lemma 1.2, each subgroup $\langle x_i\rangle$, where $i = \overline{1, r}$, is N_G^d-admissible. Lemma 1.1 implies that

$$\langle x_i\rangle \cap N_G^d \neq E \quad \text{for any} \quad i = \overline{1, r}.$$

This means that $M \cap N_G^d$ is a free Abelian subgroup of rank r with the norm N_G^d.

Theorem 1.3 is proved.

Lemma 1.3. *If the norm N_G^d of decomposable subgroups of an arbitrary group G is non-Dedekind and the group G contains a nonprimary cyclic subgroup $\langle x\rangle \times \langle y\rangle = \langle xy\rangle$ of order $p^m q^n$ (p and q are different prime numbers), then $\left\langle x^{p^{m-1}} y^{q^{n-1}}\right\rangle \subset N_G^d$.*

Proof. Let $G \supset \langle x \rangle \times \langle y \rangle$, where $|x| = p^m$, $|y| = q^n$, $m \geq 1$, $n \geq 1$, and p and q are different prime numbers. Then $\langle x, y \rangle$ is an N_G^d-admissible subgroup and, hence, the subgroups $\langle x \rangle$ and $\langle y \rangle$ are also N_G^d-admissible. Since N_G^d is a non-Dedekind group, by virtue of Lemma 1.1,

$$\langle x \rangle \cap N_G^d \neq E \qquad \text{and} \qquad \langle y \rangle \cap N_G^d \neq E.$$

Therefore, $\left\langle x^{p^{m-1}} y^{q^{n-1}} \right\rangle \subset N_G^d$.

Lemma 1.3 is proved.

The following theorem valid for both periodic and nonperiodic groups is a corollary of Lemma 1.3:

Theorem 1.4. *An arbitrary group G with the non-Dedekind norm N_G^d of decomposable subgroups contains nonprimary Abelian subgroups if and only if its norm N_G^d contains subgroups with this property.*

2. On Nonperiodic Groups with Non-Dedekind Locally Nilpotent Norm of Decomposable Subgroups

In [4], it is shown that, in the periodic case, the non-Abelian locally nilpotent groups all decomposable subgroups of which are normal (or the system of these groups is empty) are either quaternion 2-groups or $\overline{HA}_p$-groups (p-groups in which all noncyclic Abelian subgroups are normal). In the nonperiodic case, these are groups of one of the types (4) or (5) from Proposition 2 with $n > 1$. In what follows, we show that, in the class of locally nilpotent nonperiodic groups, the condition according to which the norms of decomposable subgroups are non-Dedekind leads to the normality of all decomposable subgroups of the group and, as a result, the indicated norm coincides with the group.

Theorem 2.1. *In a nonperiodic locally solvable group G, the norm N_G^d of decomposable subgroups is locally nilpotent and non-Dedekind if and only if N_G^d is a nonperiodic di-group of one of the types (4) or (5) from Proposition 2 with $n > 1$.*

Proof. The *sufficiency* of the conditions of the theorem is obvious because each group mentioned in the conditions of the theorem is a nilpotent group of class 2.

We now prove *necessity.* Let the norm N_G^d of decomposable subgroups of the nonperiodic locally solvable group G be a periodic locally nilpotent non-Dedekind subgroup. It follows from the description of di-groups in [4] that N_G^d is either a quaternion 2-group of order greater than 8 or an $\overline{HA}_p$-group (see [7]). In each of these cases, N_G^d contains a finite subgroup F normal in G. Since $[G\colon C_G(F)] < \infty$, there exists an element x of infinite order such that $x \in C_G(F)$. This implies that the subgroup $\langle x, F \rangle^{|F|} = \langle x \rangle^{|F|}$ is N_G^d-admissible. By virtue of Lemma 1.1, N_G^d must be Dedekind, which contradicts the condition.

Hence, N_G^d is a locally nilpotent nonperiodic di-group. By using the description of these groups (Propositions 1 and 2), we conclude that, among these groups, only groups of type (4) or (5) with $n > 1$ from Proposition 2 are locally nilpotent.

Theorem 2.1 is proved.

The following theorem gives the complete description of nonperiodic locally nilpotent groups whose norm N_G^d of decomposable subgroups is non-Dedekind:

Theorem 2.2. *In a nonperiodic locally nilpotent group G, the norm N_G^d of decomposable subgroups is non-Dedekind if and only if $G = N_G^d$ and G is a group of one of the following types:*

(1) $G = Q \times B$, where Q is a quaternion group and B is a torsion-free Abelian group of rank 1;

(2) $G = \langle a \rangle \leftthreetimes B$, *where* $|a| = p^n$, p *is a prime number,* $n > 1$, B *is a torsion-free incomplete Abelian group of rank 1, and* $[\langle a \rangle, B] = \left\langle a^{p^{n-1}} \right\rangle$.

Proof. The *sufficiency* of the conditions of the theorem is obvious.

We now show their *necessity.* Let G be a locally nilpotent nonperiodic group with the non-Dedekind norm N_G^d. In view of Theorem 2.1, N_G^d is a group of one of the types (1) or (2) of this theorem. Since all decomposable Abelian subgroups of the norm N_G^d are mixed, by using Theorem 1.1, we conclude that any decomposable Abelian subgroup of the group G is mixed. Hence, the periodic part $T(G)$ of the group G does not contain decomposable subgroups and, by virtue of Proposition 1, is either a locally cyclic p-group for a prime number p or a (finite or infinite) quaternion 2-group. Therefore,

$$T(G) \cap N_G^d \supset \langle a_1 \rangle, \quad \text{where} \quad |a_1| = p.$$

Since $\langle a_1 \rangle \lhd G$ and the group G is locally nilpotent, we have $\langle a_1 \rangle \subseteq Z(G)$.

We now show that any pair of infinite cyclic subgroups in the group G has a nontrivial intersection. Indeed, if $x, y \in G$, $|x| = |y| = \infty$, then the subgroups $\langle a_1, x \rangle$ and $\langle a_1, y \rangle$ are N_G^d-admissible. Hence, the subgroups $\langle x^p \rangle$ and $\langle y^p \rangle$ are also N_G^d-admissible. Since N_G^d is non-Dedekind, by using Lemma 1.1, we get

$$\langle x^p \rangle \cap N_G^d \neq E \qquad \text{and} \qquad \langle y^p \rangle \cap N_G^d \neq E.$$

Then

$$\langle x^p \rangle \cap B \neq E, \qquad \langle y^p \rangle \cap B \neq E$$

and, hence,

$$\langle x \rangle \cap \langle y \rangle \neq E.$$

This implies that $G/T(G)$ is a torsion-free locally nilpotent group without decomposable subgroups. By virtue of Proposition 1, this group is a torsion-free Abelian group of rank 1, which means that the commutant $G' \subset T(G)$.

Further, we separately consider each indicated case for the norm N_G^d.

1. Let $N_G^d = Q \times B$ be a group of type (1) from Theorem 2.1, where $Q = \langle q_1, q_2 \rangle$ is a quaternion group, $|q_1| = |q_2| = 4$, ${q_1}^2 = {q_2}^2$, ${q_2}^{-1} q_1 q_2 = {q_1}^{-1}$, and B is a torsion-free Abelian group of rank 1. Thus, $T(G)$ is a quaternion 2-group. Let $\langle q \rangle \subset T\,(G)$, where $|q| = 8$. Then $\langle q \rangle$ is a unique cyclic subgroup of order 8 in $T\,(G)$ and, hence, $\langle q \rangle \lhd G$. Without loss of generality, we can set $q^2 = q_1$. We take an element $b \in B$ of infinite order permutable with qq_2. Then $(\langle b \rangle \times \langle qq_2 \rangle)$ is an N_G^d-admissible subgroup and $\langle qq_2 \rangle$ is also N_G^d-admissible. However, $[qq_2,\, q_2] = {q_1}^{-1} = q^{-2} \notin \langle qq_2 \rangle$ and, therefore,

$$T\,(G) = T\left(N_G^d\right) = Q.$$

Let $C = C_G\,(Q)$ be a centralizer of the subgroup Q in G. Since, for each element $g \in G$ of infinite order, the subgroup $\left\langle q_1^2, g \right\rangle$ is N_G^d-admissible, $\left\langle g^2 \right\rangle$ is also N_G^d-admissible and $g^2 \in C$. This implies that $\exp\,(G/C) = 2$ and, hence, G/C is Abelian. Therefore, the commutant $G' \subseteq Q \cap C = \left\langle q_1^2 \right\rangle$. Since every decomposable subgroup of the group G contains G', all decomposable subgroups of G are normal. Hence, in this case,

$$G = N_G^d.$$

2. Let $N_G^d = \langle a \rangle \leftthreetimes B$ be a group of type (2) from Theorem 2.1, where $|a| = p^n,\ p$ is a prime number, $n > 1,$ and B is a torsion-free incomplete Abelian group of rank 1; $[\langle a \rangle, B] = \left\langle a^{p^{n-1}} \right\rangle = \langle a_1 \rangle$. Assume that $T(G) \neq \langle a \rangle$. Since $T(G)$ does not contain decomposable subgroups, there exists an element $c \in T(G)$ such that $c^p = a$ or $T(G) = \langle a, q \rangle$ is a quaternion 2-group, where $|a| = 2^n,\ n > 1,\ |q| = 4,\ q^2 = a_1,$ and $q^{-1}aq = a^{-1}$.

If $T(G)$ contains an element c such that $c^p = a$, then it follows from the condition $[\langle a \rangle, B] \neq E$ that $[a, b] = a_1$ for some element $b \in B$. Since $a_1 \in Z(G)$ and $|bc| = \infty$, we conclude that $(\langle a_1 \rangle \times \langle bc \rangle)$ is an N_G^d-admissible subgroup and, therefore,

$$[bc, b] \in G' \cap \langle bc, a_1 \rangle = \langle a_1 \rangle .$$

We set $[bc, b] = [c, b] = a_1^{\alpha}$. Thus, $[a, b] = [c^p, b] = [c, b]^p = 1$, which is impossible. Hence,

$$T(G) = T(N_G^d) = \langle a \rangle .$$

We now prove that, in this case, the commutant G' of the group G also has a prime order. To this end, it suffices to show that any nonidentity commutator of the group G has the order p. Let $x,\ y \in G$ and $[x, y] \neq 1$. Then the subgroup $H = \langle T(G), x, y \rangle$ is nilpotent and the quotient group $H/T(G)$ is cyclic. Hence,

$$H = T(G) \leftthreetimes \langle h \rangle \quad \text{for some element} \quad h \in H, \quad |h| = \infty.$$

Since $a_1 \in T(G) \cap Z(H)$, the subgroup $\langle a_1, h \rangle$ is N_G^d-admissible. Therefore,

$$[T(G), \langle h \rangle] \subset T(G) \cap \langle a_1, h \rangle = \langle a_1 \rangle, \quad |H'| = p, \quad |[x, y]| = p, \quad |G'| = p.$$

Since any decomposable subgroup of the group G is mixed, it contains a commutant G'; therefore, $G = N_G^d$ and, hence, all decomposable subgroups in G are normal.

Now let $T(G) = \langle a, q \rangle$ be a quaternion 2-group, $|a| = 2^n,\ n > 1,\ |q| = 4,\ q^2 = a_1,$ and $q^{-1}aq = a^{-1}$. In the nilpotent group $\langle T(G), x \rangle$, where $x \in G$ and $|x| = \infty$, its periodic part is non-Abelian and finite. Hence, $x^k \in C_G(\langle a, q \rangle)$ for some natural number k. Then the subgroup $\left(\left\langle x^k \right\rangle \times \langle q \rangle\right)$ is N_G^d-admissible and

$$[a, q] \in \langle a \rangle \cap \left(\left\langle x^k \right\rangle \times \langle q \rangle\right) = \langle a_1 \rangle.$$

Thus, $|a| = 4$ and $T(G) = \langle a, q \rangle$ is a quaternion group of order 8. Repeating the reasoning used in Sec. 1, we obtain

$$G' \subseteq T(G) \cap C_G(T(G)) = \langle a_1 \rangle .$$

This means that G is a di-group. Hence, $G = T(G) \times B = N_G^d$, which contradicts the condition. Thus, this case is impossible and Theorem 2.2 is proved.

Corollary 2.1. *Any nonperiodic locally nilpotent group G with non-Dedekind norm N_G^d of decomposable subgroups is a nilpotent group of class (2).*

Note that the class of nonperiodic groups with locally nilpotent non-Dedekind norm N_G^d is broader than the class of locally nilpotent nonperiodic groups with the same restrictions imposed on the norm. An example presented below shows that there exist nonperiodic groups that are not locally nilpotent in which the norm N_G^d of decomposable subgroups is a non-Dedekind nilpotent group.

Example 2.1. Consider

$$G = B \leftthreetimes \langle q_1, q_2 \rangle,$$

where B is a torsion-free Abelian group of rank 1, $|q_1| = 8$, $|q_2| = 4$, ${q_1}^4 = {q_2}^2$, ${q_2}^{-1} q_1 q_2 = {q_1}^{-1}$, $[B, \langle q_2 \rangle] = E$, and ${q_1}^{-1} b q_1 = b^{-1}$ for all $b \in B$.

The group G is not locally nilpotent but its norm $N_G^d = \langle {q_1}^2, q_2 \rangle \times B$ is nilpotent.

3. Relations between the Norms of Decomposable Subgroups and Noncyclic Abelian Subgroups in Nonperiodic Locally Solvable Groups

In this section, we consider the relations between the norms of solvable subgroups and noncyclic Abelian subgroups. For a group G, the intersection of normalizers of all noncyclic Abelian subgroups of this group (provided that the system of these subgroups is not empty) is called the *norm of noncyclic Abelian subgroups* of this group and denoted by N_G^A [8].

It is clear that if the sets of noncyclic Abelian subgroups and decomposable subgroups of the nonperiodic group G coincide, then the corresponding norms also coincide. On the other hand, the equality of the indicated norms does not imply that the sets of noncyclic Abelian subgroups and decomposable subgroups coincide.

Example 3.1. Consider a Chernikov IH-group [6, p. 176]

$$G = A \leftthreetimes \langle b \rangle,$$

where $A \supseteq A_1 \times \langle c_1 \rangle \times \langle c_2 \rangle$, A_1 is a torsion-free noncyclic subgroup of rank 1, $|b| = 2$, $b^{-1} a b = a^{-1}$ for any element $a \in A$, $|c_1| = p$, $|c_2| = q$, and p and q are different odd prime numbers. In this group, the subgroup A_1 is indecomposable and noncyclic and the subgroup $\langle c_1 \rangle \times \langle c_2 \rangle$ is decomposable and cyclic. At the same time,

$$N_G^d = N_G^A = G.$$

The examples presented below show that the inclusions $N_G^d \subset N_G^A$ or $N_G^A \subset N_G^d$ are possible in a nonperiodic locally solvable group.

Example 3.2. Consider

$$G = (((\langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_6 \rangle) \leftthreetimes \langle b \rangle) \leftthreetimes \langle c \rangle) \leftthreetimes \langle d \rangle,$$

where

$$|a_i| = \infty, \quad i = \overline{1,6}, \qquad |b| = 7, \qquad |c| = 3, \qquad |d| = 4,$$

$$b^{-1} a_i b = a_{i+1}, \quad i = \overline{1,5}, \qquad b^{-1} a_6 b = a_1^{-1} a_2^{-1} \ldots a_6^{-1},$$

$$c^{-1} a_1 c = a_2, \qquad c^{-1} a_2 c = a_1^{-1} a_2^{-1}, \qquad c^{-1} a_3 c = a_4, \qquad c^{-1} a_4 c = a_3^{-1} a_4^{-1},$$

$$c^{-1} a_5 c = a_6, \qquad c^{-1} a_6 c = a_5^{-1} a_6^{-1}, \qquad c^{-1} b c = b^2,$$

$$d^{-1} a_i d = a_i^{-1}, \quad i = \overline{1,6}, \qquad [b, d] = [c, d] = 1.$$

In this group, $N_G^A = \langle a_1, a_2, \ldots, a_6, d \rangle$. Since

$$a_i^n \notin N_G(\langle b, d \rangle) \ \text{ for } \ i = \overline{1,6}, \quad b \notin N_G(\langle a_i, d^2 \rangle), \quad c \notin N_G(\langle a_i, d^2 \rangle), \quad \text{and} \quad d \notin N_G(\langle a_1 b, d^2 \rangle),$$

we get

$$N_G^d = \langle d^2 \rangle = Z(G) \qquad \text{and} \qquad N_G^d \subset N_G^A.$$

Example 3.3. Let

$$G = \langle a \rangle \leftthreetimes B,$$

where $|a| = p$ is an odd prime number ($p \neq 2$), B is a torsion-free incomplete noncyclic Abelian subgroup of rank 1, and $G' = \langle a \rangle$.

In this group, $N_G^d = G$. Since $a \notin N_G(B)$, we conclude that $N_G^A \subset N_G^d$.

Lemma 3.1. *If the norm N_G^d of decomposable subgroups of a nonperiodic group G is non-Dedekind and all its decomposable Abelian subgroups are mixed, then $N_G^A \subseteq N_G^d$. Moreover, the case $N_G^A \neq N_G^d$ is possible.*

Proof. By virtue of Theorem 1.1, each decomposable Abelian subgroup of the group G is mixed. Hence, we have the inclusion $N_G^A \subseteq N_G^d$. As an example of a group in which $N_G^A \neq N_G^d$, we can mention the group from Example 3.3.

Lemma 3.1 is proved.

Theorem 3.1. *If the norm N_G^d of decomposable subgroups of a nonperiodic locally solvable group G is locally nilpotent and non-Dedekind, then $N_G^A \subseteq N_G^d$. Moreover, both cases $N_G^A \subset N_G^d$ and $N_G^A = N_G^d$ are realized.*

Proof. By Theorem 2.1, the norm N_G^d is a nonperiodic group of one of the types (1) or (2) from this theorem. In both cases, all decomposable Abelian subgroups of the norm N_G^d are mixed. By using Theorem 1.1, we conclude that all decomposable Abelian subgroups of the group G are also mixed and, hence, noncyclic. Therefore, the inclusion $N_G^A \subseteq N_G^d$ is true.

This inclusion is strict, e.g., in the case $G = N_G^d = Q \times B$, where $Q = \langle q_1, q_2 \rangle$ is a quaternion group of order 8 and the group B is isomorphic to an additive group of rational numbers. In this case, the subgroup contains an infinite sequence of subgroups

$$\langle b_1 \rangle \subset \langle b_2 \rangle \subset \ldots \langle b_n \rangle \subset \ldots,$$

where $|b_1| = \infty$, $b_{n+1}^{\alpha_{n+1}} = b_n$, and $(\alpha_{n+1}, 2) = 1$ for $n = 1, 2, \ldots$.

It is easy to see that the isolator A [9, p. 411] of the semigroup $\langle q_1 b_1 \rangle$ is noncyclic because we take the roots of any odd power of the element q_1. Moreover, $A \ntriangleleft G$ because $[q_2, A] = [q_2, q_1] \notin A$. Hence,

$$N_G^A = \langle q_1^2 \rangle \times B = Z(G) \neq N_G^d = G.$$

If $G = N_G^d = Q \times B$, where $Q = \langle q_1, q_2 \rangle$ is a quaternion group of order 8 and B is either a group isomorphic to an additive group of binary fractions or an infinite cyclic group, then all decomposable subgroups and all noncyclic Abelian subgroups of G are normal [10]. Hence, in this case, $N_G^A = N_G^d$.

Theorem 3.1 is proved.

Lemma 3.2. *If the conditions $N_G^A \nsubseteq N_G^d$ and $N_G^d \nsubseteq N_G^A$ are satisfied in a locally solvable nonperiodic group G, then its norm N_G^d of decomposable subgroups is Dedekind.*

Proof. Assume that the norms N_G^A and N_G^d in the locally solvable nonperiodic group G satisfy the conditions of the lemma. Then G contains a noncyclic indecomposable Abelian subgroup B which is not N_G^d-admissible and a nonprimary cyclic subgroup $\langle c \rangle$ which is not N_G^A-admissible. In this case, it is obvious that B is either a quasicyclic group or a torsion-free locally cyclic group of rank 1.

We now show that, in this group, the norm N_G^d of decomposable subgroups is Dedekind. If N_G^d is a nonperiodic group, then, in view of the fact that the subgroup $\langle c\rangle$ is N_G^d-admissible, we conclude that there exists an element x of infinite order that belongs to the centralizer $C_G(c)$. Then the Abelian subgroup $\langle c, x\rangle$, together with its characteristic subgroup $\langle c\rangle$, is N_G^A-admissible, which contradicts the choice of $\langle c\rangle$. Hence, N_G^d is a periodic locally solvable group. Moreover, if $\left|N_G^d\right| < \infty$, then, by Lemma 1.1, N_G^d is a Dedekind group.

Let N_G^d be an infinite periodic group. Assume that N_G^d does not satisfy the condition of minimality for Abelian subgroups. Then, in the group $N_G^d \cap C_G(c)$, we can select noncyclic Abelian subgroups A_1 and A_2 such that

$$(A_1 \cup A_2) \cap \langle c\rangle = E \qquad \text{and} \qquad A_1 \cap A_2 = E.$$

In this case, the subgroup $\langle A_2, c\rangle \cap \langle A_1, c\rangle = \langle c\rangle$ is N_G^A-admissible, which contradicts its choice. Hence, N_G^d is a group with condition of minimality for Abelian subgroups. By Corollary 1.1, in this case, the norm N_G^d is Dedekind.

Lemma 3.2 is proved.

Theorem 3.2. *If, in a nonperiodic locally solvable group G, at least one of the norms N_G^A or N_G^d is non-Dedekind and the norm N_G^d is infinite, then one of the inclusions $N_G^A \subseteq N_G^d$ or $N_G^d \subseteq N_G^A$ is true.*

Proof. If the norm N_G^d of decomposable subgroups of the group G is non-Dedekind, then the assertion of the theorem follows from Lemma 3.2. Therefore, in what follows, we assume that N_G^d is a Dedekind group and the norm N_G^A of noncyclic Abelian subgroups of the group G is non-Dedekind. It is easy to see that, in this case, the norm N_G^A is infinite.

We now assume that, under these conditions,

$$N_G^A \not\supseteq N_G^d \qquad \text{and} \qquad N_G^A \not\subseteq N_G^d.$$

Since the norm N_G^d is infinite, it follows from the proof of Lemma 3.2 that this norm is Dedekind and satisfies the condition of minimality. Hence, N_G^d is a finite extension of the complete subgroup P. This assumption also implies that the group G contains a nonprimary cyclic subgroup $\langle c\rangle$ which is not N_G^A-admissible and a noncyclic indecomposable Abelian subgroup B which is not N_G^d-admissible. Moreover, B is either a quasicyclic group or a torsion-free locally cyclic group of rank 1.

Further, we separately consider each of the above-mentioned cases for the subgroup B.

1. Let B be a quasicyclic subgroup. We show that, in this case, it is a maximum Abelian subgroup of the group G. Indeed, otherwise, in G, there exists a nonidentity subgroup $\langle g\rangle$ such that $B \cap \langle g\rangle = E$ and $[B, \langle g\rangle] = E$. Then the subgroup $B \times \langle g\rangle$ is N_G^d-admissible. Moreover, if $|g| = \infty$, then the subgroup B, as a characteristic subgroup of the group $B \times \langle g\rangle$, is also N_G^d-admissible, which contradicts its choice. If $|g| < \infty$, then the subgroup $\langle B, g\rangle^{|g|} = B$ is also N_G^d-admissible. Hence, B is the maximum Abelian subgroup of the group G and $B \ntriangleleft G$.

Applying Corollary 1.3 in [6] to the group $G_1 = BN_G^d$ and taking into account the fact that N_G^d is a finite extension of the complete subgroup P, we conclude that G_1 is a group with the condition of minimality for Abelian subgroups. Since B is a maximum Abelian subgroup, $B = P \triangleleft N_G^d$, which contradicts the choice of B. Hence, B cannot be a quasicyclic group.

2. We now consider the case where B is a torsion-free locally cyclic group of rank 1.

Since the subgroup $\langle c\rangle$ is not primary and not N_G^A-admissible, at least one of its Sylow subgroups is also not N_G^A-admissible. Let $\langle c\rangle_p$, where p is a prime number, be a subgroup of this type. If $p \notin \pi(P)$ or P is nonprimary, then, in P, we can select a quasicyclic q-subgroup P_1 with $q \neq p$ and $\langle c\rangle_p P_1 = \langle c\rangle_p \times P_1$, where $\langle c\rangle_p$ is an N_G^A-admissible subgroup, which contradicts its choice. Hence, $\langle c\rangle_p P$ is a p-group.

If $\langle c\rangle_p \subset P$, then $\langle c\rangle_p$ is a subgroup of a quasicyclic p-group and, hence, it is also N_G^A-admissible, which is impossible. Let $\langle c\rangle_p \not\subset P$. If the complete p-group P is not quasicyclic, then $\langle c\rangle_p P$ contains an elementary Abelian subgroup of order p^3. In this case, there exists a subgroup $\langle a_1\rangle \times \langle a_2\rangle$ of order p^2 such that

$$\langle c\rangle_p \cap \langle a_1, a_2\rangle = E.$$

Then the subgroup $\Big(\langle a_1\rangle \times \langle c\rangle_p\Big) \cap \Big(\langle a_2\rangle \times \langle c\rangle_p\Big) = \langle c\rangle_p$ is N_G^A-admissible, which contradicts its choice. Hence, P is a quasicyclic p-group and there exists an element $a \in \langle c\rangle_p P$ of order p such that the subgroup $\langle a\rangle \times \langle c\rangle_p$ is N_G^A-admissible.

Moreover, if the subgroup N_G^A is nonperiodic, then there exists an element $x \in N_G^A$ such that

$$|x| = \infty \qquad \text{and} \qquad \Big[\langle x\rangle, \langle c\rangle_p\Big] = E.$$

Therefore, the subgroup $\langle x\rangle \times \langle c\rangle_p$ and, hence, also the subgroup $\langle c\rangle_p$ are N_G^A-admissible. We again arrive at a contradiction. Thus, N_G^A is a periodic group. Therefore, in the group $N_G^A B = N_G^A \times B$, each subgroup of the norm N_G^A is normal and, hence, N_G^A is Dedekind, which contradicts the condition.

Theorem 3.2 is proved.

The examples presented below show that the condition of infinity of the norm N_G^d in Theorem 3.2 is essential.

Example 3.4. Let

$$G = (\langle a\rangle \leftthreetimes B) \leftthreetimes \langle c\rangle\,,$$

where $|a| = p$ is a prime number ($p \neq 2$), B is a group isomorphic to an additive group of q-adic fractions, $q \notin \{2, p\}\,,\ B = B_1\langle x\rangle,\ x^2 \in B_1,\ x^{-1}ax = a^{-1},\ [B_1\,\langle, a\rangle] = E,\ |c| = 2,\ [c, a] = 1,$ and $c^{-1}bc = b^{-1}$ for any element $b \in B$.

In this group, all periodic decomposable subgroups have the order $2p$ and are groups of the form

$$\left\langle a^m c b_1^k \right\rangle,$$

where $b_1 \in B_1,\ k \in \{0, 1\}\,,$ and $(m, p) = 1$. Thus, all nonperiodic decomposable subgroups are mixed and contained in the group $B_1 \times \langle a\rangle$ and, hence, they are normal in G. Since $N_G\left(\left\langle a^m c b_1^k\right\rangle\right) = \left\langle a^m c b_1^k\right\rangle$, we conclude that $N_G^d = \langle a\rangle$.

We now determine the norm N_G^A of noncyclic Abelian subgroups of the group G. It is obvious that G does not contain periodic noncyclic Abelian subgroups but all mixed Abelian subgroups contain $\langle a\rangle$ and are subgroups of the group $(B_1 \times \langle a\rangle)$. It is easy to see that all these subgroups are normal in G. Further, all noncyclic Abelian subgroups of rank 1 are contained either in the subgroup B or in the subgroups $g^{-1}Bg,\ g \in G$, conjugate to this subgroup, or in the group $(B_1 \times \langle a\rangle)$. Consider an infinite sequence of subgroups in B_1:

$$\langle b_1\rangle \subset \langle b_2\rangle \subset \ldots \langle b_n\rangle \subset \ldots\,,$$

$$|b_1| = \infty, \quad b_{n+1}^{\alpha_{n+1}} = b_n, \quad \alpha_{n+1} \in \mathbb{N}, \quad \text{and} \quad (\alpha_{n+1}, p) = 1 \quad \text{for} \quad n = 1, 2, \ldots.$$

It is easy to see that the isolator of the subgroup $\langle ab_1\rangle$ is noncyclic because we take the root of the element a of any power mutually prime with p. Moreover, $N_G(A) = \langle a, B_1\rangle$. Since $N_G(B) = B \leftthreetimes \langle c\rangle$, we conclude that

$$N_G^A = B_1 \qquad \text{and} \qquad N_G^d \cap N_G^A = E.$$

Example 3.5. Let

$$G = (\langle a \rangle \leftthreetimes B) \leftthreetimes \langle c \rangle\,,$$

where $|a| = p$ is a prime number $(p \neq 2)$, B is a group isomorphic to an additive group of p-adic fractions, $B = B_1\langle x \rangle$, $x^2 \in B_1$, $x^{-1}ax = a^{-1}$, $[B_1\,\langle, a \rangle] = E$, $|c| = 2$, $[c, a] = 1$, and $c^{-1}bc = b^{-1}$ for any element $b \in B$.

As in Example 3.4, in this group, the norm of decomposable subgroups $N_G^d = \langle a \rangle$. At the same time, the norm of noncyclic Abelian subgroups $N_G^A = (B_1 \leftthreetimes \langle c \rangle)$. This result follows from the fact that, for any nonidentity element $y_1 \in B_1$, the isolator of the subgroup $\langle ay_1 \rangle$ is cyclic and, hence, the element c normalizes any noncyclic Abelian subgroup of the group G. In this case, the norm N_G^A of noncyclic Abelian subgroups is non-Dedekind and $N_G^d \cap N_G^A = E$.

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